VOL. 4, NO. 10, DECEMBER 2009 ISSN 1819-6608
ARPN Journal of Engineering and Applied Sciences

www.arpnjournals.com

ROBUST MINIMUM VARIANCE CONTROLLER USING OVER-PARAMETERIZED CONTROLLER

M. M. Al-Imam[1] and M. M. Mustafa[2]
[1]University of Kuala Lumpur, Section 14, Jalan Teras Jernang, B. B. Bangi, Selangor, Malaysia
[2]Universiti Kebangsaan Malaysia, UKM Bangi, Selangor, Malaysia
E-Mail: alemam_m@yahoo.com

ABSTRACT

In this article, a robust, stable and fast calculable controller that reduces the variance to the minimum for minimal and non-minimal phase Linear Time Invariant (LTI) system is proposed. The calculation is based on an algorithm that overcomes the complexity of conventional methods. The algorithm utilizes Diophantine Equation to obtain over-parameterized transfer function polynomial forms that contribute to the variance reduction. It analytically proves that increasing the order of the parameterized controller's coefficients makes the variance converge to the minimum, while maintaining the same simplicity of calculation. Simulated examples for different LTI models support our findings.

Keywords: output variance, over-parameterization, pole-placement.

1. INTRODUCTION

Almost all practical control systems are often affected by various external disturbances [14, 7, 15], which can deteriorate the performance of feedback control Linear Time Invariant (LTI) systems. Previous works in the field have addressed the various aspects of disturbance rejection such as optimal disturbance rejection [8], asymptotic disturbance rejection [18], optimal disturbance rejection using PID controller [13], disturbance rejection with saturated actuator [9], and practical application of altitude tracking and disturbance rejection [2].

In the domain of disturbance rejection, Minimum Variance (MV) was early introduced in the valuable work of Astrom and Wittenmark [1]. Another detailed account was written in 2006 by Landau and Zito [15]. Minimum Variance regulation only applies to plants having a discrete-time model with stable zeros. Generalized Minimum Variance (GMV) came to the scene in treating plants with unstable zeros. It is an extension to the existent minimum variance strategy [15]. Pioneers have developed the technique of Generalized Minimum Variance (GMV) [3 and 4]. Several studies have implemented the GMV technique, such as self-tuning PID controller based [19] and a self-tuning controller for minimum and non-minimum phase systems [16]. Ertunc *et al.*, [6] did a comparative study between the performance of GMV and PID controller. However, the main limitation of this technique is that the existence of a solution to a stable closed loop system is not guaranteed especially in the case of several unstable zeros. Thus, the solution is always an asymptotically stable closed loop system [15, 12].

These findings motivated other researchers to develop LQG feedback control [17, 12]. However, LQG control scheme exhibits heavy calculation load [17] in addition to no guarantee to the conventional desired gain and phase margin [11].

Our objective in this work is to develop a controller that features fast calculated coefficients and robustness against system time delay and pole locations. The proposed controller can perform reduced variance without the need to factorize the zeros.

The concept of over-parameterization is adopted to ensure variance reduction. Earlier works have utilized a similar concept [10]. It consists of developing a single structure controller that track reference signal of LTI system in addition to achieving variance reduction on the output. Halpern [10] has always assumed a unity characteristic equation and heuristically, his developed algorithm reduces the output variance after a careful tuning to a weighting matrix. Davies and Zarrop [5] worked on over-extending the minimal solution of pole placement, using Diophantine Equation, in order to achieve variance reduction of LTI system. They analytically proved that over-extending the parameter of its proposed controller to infinity, the variance decreases to a variance of an LQG controller [5].

Our approach to the problem is to adopt a Diophantine equation in order to calculate the over-parameterized coefficients for the controller of an ARMAX model. The minimum variance for a general system output that may include unstable poles is analytically derived. The result is truncated so that it leads to dual Diophantine Equations, that both can be over-parameterized. This step is essential in overcoming complexity of calculating the H_2 norm of the minimum variance. The calculated polynomial that minimizes the H_2 norm of the first over-parameterized polynomial also minimizes H_2 norm of the second, which leads to the solution. The proof is realized by introducing a sub-optimal solution and analytically proving that its over-parameterized H_2 norm converges to zero. This causes the optimal solution to converge to zero as well, because the optimal H_2 norm is sandwiched by the sub-optimal norm from above.

ARPN Journal of Engineering and Applied Sciences

www.arpnjournals.com

2. PRELIMINARIES ON VARIANCE MINIMIZATION

A linear single-input single-output system represented by an autoregressive moving average with auxiliary input ARMAX model of the form

$$A(q^{-1})y(t) = q^{-d}B(q^{-1})u(t) + C(q^{-1})e(t) \qquad \ldots(1)$$

where,

$$A(q^{-1}) = 1 + a_1 q^{-1} + \ldots + a_{n_a} q^{-n_a}$$

$$B(q^{-1}) = b_0 + b_1 q^{-1} + \ldots + b_{n_b} q^{-n_b}$$

$$C(q^{-1}) = c_0 + c_1 q^{-1} + \ldots + c_{nc} q^{-nc}$$

Polynomials A (q-1) and B (q-1) in equation (1) are the denominator and numerator of the ARMAX system respectively. The unit delay is represented by the variable q^{-d}. The command signal, u (t), is the control signal that excites the ARMAX model, described in the discrete time t. The output of the system is denoted as y (t). The system is described in discrete time domain t. a source of white spectrum source is defined with the random signal, e (t).

A general controller for regulation is defined in equations (2), (3) and (4) with the two polynomials G (q-1) and F (q-1) that denote the controller transfer function numerator and denominator. Figure-1 presents the general form of ARMAX model with noise source, e (t) and the backward shift operator, q, is replaced with its equivalent *z* operator.

$$F(q^{-1})u(t) = G(q^{-1})H(q^{-1})r(t) - G(q^{-1})y(t) \qquad \ldots (2)$$

Where

$$F(q^{-1}) = 1 + f_1 q^{-1} + \cdots + f_{n_f} q^{-n_f} \qquad \ldots (3)$$

$$G(q^{-1}) = g_0 + g_1 q^{-1} + \cdots + g_{n_g} q^{-n_g} \qquad \ldots (4)$$

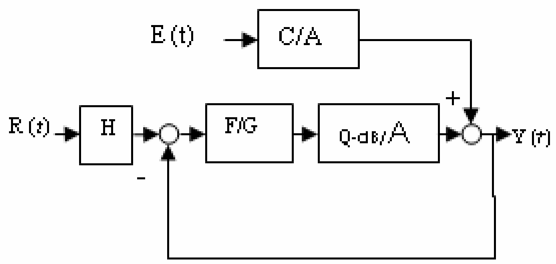

Figure 1: ARMAX model

The feed-forward polynomial H (q-1) is normally chosen to give a good set-point tracking. In this paper, H (q-1) is normally chosen so that the overall DC gain for the closed loop system is unity.

Let the polynomial T (q-1) denotes the characteristics equation of the sys- tem (i.e. the denominator of the closed loop transfer function). Therefore, T (q-1) is described according to the following Diophantine equation

$$T(q^{-1}) = A(q^{-1})F(q^{-1}) + q^{-d}B(q^{-1})G(q^{-1}) \qquad \ldots (5)$$

Equation (5) has a unique solution if

$$n_f = n_d + k - 1 \text{ and } n_g = n_a - 1 \qquad \ldots (6)$$

Let F_0 (*q*-1) and G_0 (*q*-1) be the minimal order for the controller satisfying equations (5) and (6). The set of all controllers that satisfy equation (5) is given by

$$F = F_0 + q^{-d}BP_{n_p} \qquad \ldots (7)$$

$$G = G_0 - AP_{n_p} \qquad \ldots (8)$$

The operator q-1 has been omitted for simplifying writing the equations. Polynomial P_{np} in equations (7) and (8) denotes a polynomial of degree n_P that we used to extend the original solution of F (q-1) and G (q-1). Substituting equations (7), (8), and (5) in (1) and (2)

$$y(t) = \frac{q^{-d}BGH}{T}r(t) + \left(\frac{F_0 + q^{-d}BP_{n_P}}{T}\right)Ce(t) \qquad \ldots (9)$$

$$u(t) = \frac{AH}{T}r(t) + \left(\frac{G_0 - AP_{n_P}}{T}\right)Ce(t) \qquad \ldots (10)$$

From equations (9) and (10), the input and the output variance are given by

$$var_y = E\left(\frac{F_0 + q^{-d}BP_{n_P}}{T}Ce(t)\right)^2$$

$$= \left\|\frac{F_0 + q^{-d}BP_{n_P}}{T}\right\|_{L_2}^2 \qquad \ldots (11)$$

$$var_u = E\left(\frac{G_0 - AP_{n_P}}{T}\right)^2 Ce(t)$$

$$= \left\|\frac{G_0 - AP_{n_P}}{T}\right\|_{L_2}^2 \qquad \ldots (12)$$

The objective is to minimize a cost function that defines the output variance, described in equation (11). The output variance is a function of the coefficients of P_{n_P}. A description of a cost function is described as

$$J = \left\|\frac{F_0 + q^{-d}BP_{n_P}}{T}\right\|_{L_2}^2 \qquad \ldots (13)$$

[5] proposed a technique that computes the optimum P_{n_P}, but the computation is not tractable because it requires the computation of mathematical expectation.[10] proposed a simpler method to reduce the output variance in equation (13) by letting equation (5) equals to one. This proposition can serves only certain cases.

The minimization of the cost function in equation (13) fulfills the objective if the reference signal of the closed loop system equals to zero. In this paper, we deal

ARPN Journal of Engineering and Applied Sciences

with the minimization of the output variance in equation (13). Precisely, we deal the minimizing of the norm of the nominal control sensitivity transfer function, whether the actual plant is minimum or non-minimum phase.

3. VARIANCE MINIMIZATION AND OVER-PARAMETERIZATION

Let polynomial B be factorized as $B = B_+B_-$, where B_+ and B_- denotes the stable and unstable factors of the polynomial respectively. Let B_-^+ to denote the normalized reciprocal of B (i.e. $B_-^+ = q^{-n_b+1}B^*/b_0$)

Lemma 1: The stable rational transfer function P_{nP} that minimizes equation (13) is given by

$$P_{n_P} = -\frac{Q}{B_+B_-^+C} \qquad \dots (14)$$

Where polynomial Q is obtained from the Diophantine Equation defined

$$RT + q^{-d}QB_- = F_0B_-^+C \qquad \dots (15)$$

Where,

$$R(0) = 1, \qquad deg(R) = \deg(B_-) + d - 1$$

And

$$\deg(Q) = \deg(T) - 1$$

Proof

Expanding the expectation function equation (11) into two terms and decomposing the first term of the result into stable and causal, and unstable and non-causal terms.

$$\left(\frac{F_0 + q^{-d}BP_{n_P}}{T}\right)C =$$

$$\frac{q^{-d}B_-}{B_-^+}\left(\frac{CF_0B_-^+}{q^{-d}B\_T} + \frac{B_+B_-^+CP\_n_P}{T}\right) \qquad \dots (16)$$

Where

$$\frac{F_0B_-^+C}{q^{-d}B\_T} = \frac{Q}{T} + \frac{R}{q^{-d}B_-} \qquad \dots (17)$$

This decomposition requires the solution of equation (15).

It will be proved later the minimum output variance is a function of R. Using equation (17), equation (16) is rewritten as

$$\left(\frac{F_0 + q^{-d}BP_{n_P}}{T}\right)C =$$

$$\frac{q^{-d}B_-}{B_-^+}\left(\frac{Q}{T} + \frac{R}{q^{-d}B_-} + \frac{B_+B_-^+CP_{n_P}}{T}\right) \qquad \dots (18)$$

Introducing $W = (Q + B\_ + B_-^+CP_{n_P})/T$. Using equation (18) with the definition of W and arranging $K = B\_/B_-^+$, the cost function of equation (13) is expressed

$$J = \left\|\frac{q^{-d}B_-}{B_-^+}\left(W + \frac{R}{q^{-d}B_-}\right)\right\|_{L_2}^2$$

$$= \frac{1}{2\pi j}\oint_{|z|=1} KK^*\left(WW^* + \frac{RW^*}{z^{-d}B_-} + \frac{R^*W}{z^dB_-^*} + \frac{RR^*}{B_{-B_-^*}}\right)\frac{dz}{z} \qquad \dots (19)$$

After interchanging the operator q with z, it is important to notice that $KK^* = \frac{B\_B_-^*}{B_-^+B_-^{+*}} = \kappa$ that is a constant. Therefore the term KK^* in equation (19) can be taken outside the integration. Since stable solution of P_{n_P} is sought in equation (18), W must be stable, hence W^* has all its poles outside the unit circle. Applying *Cauchy Theorem*, equation (19) becomes

$$J = \frac{1}{2\pi j}KK^*\oint_{|z|=1}\left(WW^* + \frac{RR^*}{B\_B_-^*}\right)\frac{dz}{z} \qquad \dots (20)$$

The first term in the integration is function in P and independent of the second term. Therefore, minimizing equation (20) is achieved by letting W equals to zero, which yields to describe P_{n_P} that minimizes the cost function of equation (13) as depicted in equation (14). The minimum cost function is then described as

$$J_{min} = \frac{1}{2\pi j}\oint_{|z|=1}\frac{B\_B_-^*}{B_-^+B_-^{+*}}\frac{RR^*}{B\_B_-^{+*}}\frac{dz}{z}$$

$$= \left\|\frac{R}{B_-^+}\right\|_{L_2}^2 \qquad (21)$$

Considering applying the control feedback system, $u(t) = -Q/Ry(t)$, where polynomials Q and R are the solutions of equation (15). According to [1], the polynomials Q and R are the solution for the General Minimum Variance problem GMV. Using equation (9) and the result of equation (21), the closed loop system after the application of high order pole-placement controller becomes

$$y(t) = \frac{q^{-d}BH}{T}r(t) + \frac{R}{B_-^+}e(t) \qquad \dots (22)$$

4. REDUCED VARIANCE USING OVE-PARAMETERIZED CONTROLLER

In order to simplify the calculation, [10] neglects the effect of the characteristics equation, T, in equation (11) and minimized the cost function

$$J = \left\|\left(F_0 + q^{-d}BP_{n_P}\right)C\right\|_{L_2}^2 \qquad \dots (23)$$

Equation (23) can be described as a sum of coefficients squares of $(F_0 + q^{-d}BP_{n\_p})C$. [10] Argued heuristically that the proposed method reduces also the output variance. However, by neglecting the effect of polynomial T, the method works only for restricted cases

ARPN Journal of Engineering and Applied Sciences

www.arpnjournals.com

when T = 1. The difficulty for other choice of T is that the formulation of equation (11) into a set of linear equations in term of unknown coefficients of polynomial as proved by [20] is computationally cumbersome. The developed method here retains simplicity comparing with the algorithm proposed by [10] and overcoming its limitation. From equation (21), the cost function in equation (13) is minimized when

$$\left(\frac{F_0 + q^{-d}BP_{n_F}}{T}\right)C = \frac{R}{B_-^+} \quad \text{… (24)}$$

Now, letting

$$\frac{R}{B_-^+} \simeq \zeta_{n_\zeta} \quad \text{… (25)}$$

Where ζ_{n_ζ} is an n_ζ order polynomial and $\zeta_{n_\zeta}(0) = 1$. It is important to recall that B_-^+ is the normalized reciprocal of the unstable part of polynomial B. Because the derivation of the proposed controller is relying on this fact in equations (24) and (25), this leads to the proposed controller is applicable to non-minimal phase systems. Equation (25) is an approximation of the long division of equation (24). The output variance can be computed as the sum of square of $\zeta_{n_\zeta}(q^{-1})$ coefficients. Obviously, it is desired to minimize these coefficients because this will result in minimum output variance. Specifically, finding the coefficients of P_{n_F} that minimizes equation (11), leads also to minimize the norm $\left\|\zeta_{n_\zeta}(q^{-1})\right\|_{L_2}^2$. Using equation (25), equation (24) becomes a Diophantine Equation.

$$T\zeta_{n_\zeta} - q^{-d}BP_{n_F c} = F_0 C \quad \text{… (26)}$$

Equation (26) can be reconfigured using Sylvester's Theorem into a set of linear equations illustrated in equation (27) and with a unique solution described in equation (28)

$$Cx = b \quad \text{… (27)}$$

$$n_P = n_T - 1 \text{ and } n_\zeta = n_b + n_c - 1 \quad \text{… (28)}$$

The minimal order solution is of no interest since is over extended by the coefficients of P_{n_F}. If n_P and n_ζ is chosen bigger than equation (28), then the solution of equation (27) is no longer unique because number of variables are more than number of equations. It is remarkable that once n_P is selected, n_ζ can be selected from $n_\zeta = n_p + n_C - n_T$. This extra degree of freedom can be exploited to reduce $\left\|\zeta_{n_\zeta}(q^{-1})\right\|_{L_2}^2$. The design problem can be formulated as a simple convex optimization problem to choose n_P greater than the value set in equation (28) and minimizing subjected to the criteria in equation (27). A proposed algorithm for synthesizing the controller is introduced.

Let F_0 and ζ_0 are the minimal order solution to equation (26). Therefore, the general solution using Diophantine equations is defined by

$$P_{n_F} = F_0 + T\phi \quad (29)$$

$$\zeta_{n_\zeta} = \zeta_0 + B'C\phi \quad (30)$$

Polynomial B_0 represents $q^{-d}B$ and polynomial ϕ is a polynomial with arbitrary coefficients. According to equations (24) and (25), it is clear that choosing ϕ, which minimizes (30) leads to the norm minimization of (29). The minimization mechanism is adopted in a similar approach of [10]; we obtain the least square solution of the equations that are result from equating coefficients of similar power in (30). More precisely, by differentiating all the coefficients of ζ_{n_ζ} and find ϕ that makes the differentiated terms equals to zero. Defining the polynomial

$$\zeta_{n_\zeta}(q^{-1}) = \sum_{i=1}^{n_\zeta} \lambda_i q^{-i}$$

$$= \zeta_0 + B'C\phi \quad (31)$$

The dependency on operator q^{-1} is omitted for simplicity and $B' = q^{-d}B$. Expanding the sum of equation (31) yields

$$\left\|\zeta_{n_\zeta}(q^{-1})\right\|_{L_2}^2 = \lambda_0^2 + \lambda_1^2 + \cdots + \lambda_{n_\lambda}^2 \quad \text{… (32)}$$

The solution for ϕ that minimizes equation (32) is expressed

$$2\lambda_i \nabla_{\phi_j} \lambda_i = 0 \quad \text{… (33)}$$

where $i = [0, n_\zeta]$ and $j = [0, n_\phi]$

Let ϕ_0 denotes the solution of equation (33) and $\zeta_{n_\zeta}^0(q^{-1})$ denotes the polynomial $\zeta_{n_\zeta}(q^{-1})$ after substituting ϕ^0 in equation (30).Therefore, polynomial $\zeta_{n_\zeta}^0(q^{-1})$ is considered the minimal norm in L_2 space and the following bound is valid

$$\left\|\zeta_{n_\zeta}^0\right\|_{L_2}^2 \le \left\|\zeta_{n_\zeta}^1\right\|_{L_2}^2 \quad \text{… (34)}$$

Where $\zeta_{n_\zeta}^1$ is a sub-optimal solution to be chosen later and equation (34) is interpreted that the norm of any suboptimal solution is greater or equal to the norm of the optimal solution for any order of n_ζ. The importance of equation (34) will be clarified in the next section.

4.1 Analytical proof for variance reduction

In this section, we introduce an analysis that describes the optimal solution of the norm in equation (30) as a function of degree n_ϕ.

Lemma 2

The norm in L_2 for the over-parameterized ζ_{n_ζ} suggested in (32) converges towards the minimum variance as the order n_ζ increases towards infinity, regardless of the characteristics equation T in (5).

Proof

Defining the fraction R/B_-^+ from equation (24) as a summation series denoted

$$J_{min} = \left\|\frac{R}{B_-^+}\right\|_{L_2}^2$$

$$= \|\textstyle\sum_{n=1}^{\infty} \alpha_i\|_{L_2}^2 \qquad \dots (35)$$

Where α_i are real coefficients and; $\alpha_0 = 1$ therefore, one concludes

$$\left\|\frac{R}{B_-^+}\right\|_{L_2}^2 \geq 1 \qquad \dots (36)$$

The objective is to develop polynomial $\zeta_{n_\zeta}(q^{-1})$, which is defined as in equation (30) that has its norm L_2 approaches the optimal norm, $\zeta_{n_\zeta}^0(q^{-1})$ of equation (34). Polynomial ϕ is determined using least square method. Adopting an approach inspired from the work of [5], we define a suboptimal solution of the problem. According to (31), let

$$\frac{\zeta_0(q^{-1})}{B'(q^{-1})C(q^{-1})} = \sum_{i=0}^{\infty} w_i q^{-i} \qquad \dots (37)$$

Arranging $B'(q^{-1})C(q^{-1}) = D(q^{-1})$ and interchanging the operator q with the discrete operator z. The infinity series in equation (37) can be represented by a finite and infinity series respectively.

$$\zeta_0 = DW_{n-1} + z^{-n}V_n \qquad \dots (38)$$

$$\frac{V_n}{D} = \sum_{i=n}^{\infty} w_i z^{-(i-n)} \text{ and}$$

$$W_{n-1} = \sum_{i=0}^{n-1} w_i z^{-i} \qquad (39)$$

Substituting equations (38) and (39) into equation (31) with a suboptimal solution $P_{n_p}^1(z^{-1}) = W_{n_p}$ in equation (29) yields

$$\zeta_{n^1_\zeta}^1(z^{-1}) = z^{-(n_p+1)} V_{n_p+1}$$

$$= z^{-(n_p+1)} D(z^{-1}) \sum_{i=n_p+1}^{\infty} w_i z^{-(i-n_p-1)} \qquad \dots (40)$$

The term, n_ζ^1, is the order of the polynomial $\zeta_{n_\zeta^1}^1$. It is important to highlight that ζ_0 is a stable polynomial. From equation (38), it is clear that the norm L_2 of $\zeta_{n_\zeta^1}^1$ is a function in n_p. Investigating the norm of $\zeta_{n_\zeta^1}^1$ in L_1 and find the relationship with n_p leads to understand the behavior of the norm of $\zeta_{n_\zeta^1}^1$ in L_2. Defining

$$\gamma = \max\{|z| : z^{n_D} D(q^{-1}) = 0\} \qquad \dots (41)$$

Let $n = n_p + 1$ and evaluating the term $\left|\frac{V_n(q^{-1})}{D(q^{-1})}\right|$ for $|q| = 1$ using *Cauchy integral* of radius γ' centered on the origin, satisfying $1 > \gamma' > \gamma$.

$$\left|\frac{V_n(z^{-1})}{D(z^{-1})}\right| = \left|\frac{1}{2\pi j}\int_{|z|=\gamma'} \frac{\sum_{i=n}^{\infty} w_i z^{-(i-n)}}{z-1}\right| \qquad \dots (42)$$

Recalling that

$$|\oint f(z)\,dz| \leq \max_{z\in C}(|f(z)|)L_c \qquad \dots (43)$$

Where, L_C is the path along the integration taken. The path in our analysis is the circumstances of the unity circle.

$$\left|\frac{V_n(z^{-1})}{D(z^{-1})}\right| \leq \max_{|z|=\gamma'}\left\{\left|\frac{\sum_{i=n}^{\infty} w_i z^\wedge - (i-n)}{z-1}\right|\right\}\gamma' \qquad (44)$$

$$\left|\frac{V_n(z^{-1})}{D(z^{-1})}\right| \leq \frac{(\gamma')^{n+1}}{\gamma'-1}\max_{|z|=\gamma'}\left\{\left|\sum_{i=n}^{\infty} w_i z^{-i}\right|\right\} \to 0 \qquad n\to\infty \qquad \dots (45)$$

From equation (38), we can relate the term $\left|\frac{V_n(z^{-1})}{D(z^{-1})}\right|$ as a function of $\zeta_0(z^{-1})$ and $W_{n-1}(z^{-1})$.

$$\frac{V_n(q^{-1})}{D(z^{-1})} = z^n\frac{\zeta_0(q^{-1})}{D} - z^n W_{n-1} \qquad \dots (46)$$

According to equation (46) and taking into account (37), equation (45) is reformulated as

$$\left|\frac{V_n(q^{-1})}{D(z^{-1})}\right| \leq \frac{(\gamma')^{n+1}}{\gamma'-1}\max_{|z|=\gamma'}\left\{\frac{\zeta_0(z^{-1})}{D(z^{-1})}\right\} \to 0 \qquad n\to\infty \qquad (47)$$

Because the term $W_{n-1}(z^{-1})$ is a finite summation and its Cauchy integral equals to zero. Taking the norm in L_2 of equation (47) and substituting it in equation (40)

$$\left\|\zeta_{n^1_\zeta}^1(z^{-1})\right\|_{L_2}^2 \leq \frac{(\gamma')^{2(n+1)}}{(\gamma'-1)^2}\max_{|z|=\gamma'}\left|\frac{\zeta_0(z^{-1})}{D(z^{-1})}\right|^2 D^2(z^{-1}) \to 0$$

$$\text{as} \qquad n\to\infty \qquad (48)$$

Equation (48) proves that the norm of the sub-optimal solution tends to zero as the order of n tends to infinity. Recalling the lower bound in equation (34), as the norm of sub-optimal solution goes to zero, the norm of the optimal solution tends to zero as well as n tends to infinity. Consequently, the optimal solution tends to zero also for the same n_P. A suitable algorithm that is used to design the higher order controller is illustrated.

Algorithm

Step 1 Select T to give the desired servo behavior

Step 2 Solve the pole-placement equation (5) to obtain the minimal order solution F_0 and G_0.

Step 3 Solve for the minimal P_0 and ζ_0 using equation (26)

Step 4 Select the desired n that will give $n_\phi = n_P - n_T$

Step 5 Find ϕ that minimizes $\|\zeta_0 + B'C\phi\|_{L_2}^2$ according to equations (310, (32), and (33).

Step 6 Obtain F_{n_P} using equation (29)

Step 7 Form the higher order controller using equation (9).

It is important to note that the spectral factorization and explicit formulation on the time delay are not required in this algorithm. Once P_0 and ζ_0 are computed, only steps four to seven are required to be repeated if different higher order controller is needed to improve the variance reduction. This feature is useful for an adaptive scheme of the on-line supervisor is used to tune the order of the controller to get the desired variance reduction.

5. RESULTS AND DISCUSSIONS

In this section, two simulated examples are introduced to demonstrate the variation of the variance as a function with the order of polynomial $\phi(q^{-1})$ Firstly, an example to consider the design of a pole placement to a plant described with a non-minimum phase and a second example demonstrates the implementation of the algorithm with a non-minimum-phase system model. The two models are configured as ARMAX models

5.1. Non-minimal phase model

A non-minimal ARMAX model is introduced. The locations of the zeros are 0, 3, and 0.201. The set of characteristics equation is in (50). The result of the measured output variance using the proposed algorithm is compared with the work of Halpern [10].

$$Y(z^{-1}) = \frac{0.603z^{-3} - 3.201z^{-2} + z^{-1}}{-0.028z^{-3} + 0.269z^{-2} - 0.98z^{-1} + 1}U(z^{-1}) + \frac{1}{-0.028z^{-3} + 0.269z^{-2} - 0.98z^{-1} + 1}E(z^{-1}) \quad \ldots (49)$$

The characteristics equation proposed

$$T(z^{-1}) = 1 - 0.9z^{-1} \quad \ldots (50)$$

The minimal degree solution of $F(q^{-1})$ and $G(q^{-1})$ are denoted by $F_0(q^{-1})$ and $G_0(q^{-1})$ respectively and are found using the Diophantine Equation (5)

$$F_0(z^{-1}) = -51.809z^{-2} + 17.246z^{-1} + 1$$
$$G_0(z^{-1}) = -2.4057z^{-2} + 13.4621z^{-1} - 17.1676 \quad \ldots (51)$$

Calculating the output variance of the system with minimal order solution using $\left\|\frac{F_0}{T}\right\|_{L_2}^2$ is 330.335. Choosing n_ϕ equals to 0, 1, 4, 9, and 19 to form high order controller as in Table-1, results with the following output variance calculation

Table-1. Variance of zero responses with increasing n_P.

n_P	n_ϕ	$\left\|\frac{F_0 + q^{-d}BP_{n_P}}{T}C\right\|_{L_2}^2$	$\left\|\frac{R}{B_-^+}\right\|_{L_2}^2$
1	0	8.853	1.0423
2	1	2.6109	1.0423
5	4	2.3982	1.0423
10	9	2.3982	1.0423
20	19	2.3982	1.0423

From Table-1, it is noticed that increasing the degree of F_{n_P} decreases the variance of the system's output towards the minimal (i.e. minimum variance of 1)

Adopting the algorithm that was developed by [10] for the same system with an initial selection of characteristics equation's poles

Table-2. Variance of zero response with the algorithm proposed by Halpern [10].

n_P	n_ϕ	$\left\|\frac{F_0 + q^{-d}BP_{n_P}}{T}C\right\|_{L_2}^2$	$\left\|\frac{R}{B_-^+}\right\|_{L_2}^2$
1	0	6.928	2.4
2	1	9.1262	2.4
5	4	11.9942	2.4
10	9	12.5732	2.4
20	19	12.4657	2.4

The measure of the variance according to the order of n_P is presented in Table-2.using the algorithm proposed by [10]. Because of a non-unity characteristics equation, the variance, in Table-2, is not linearly collated with order of the order of n_P.

The calculation of minimum variance, from equation (21) $J_{min} = \left\|\frac{R}{B_-^+}\right\|_{L_2}^2$ = 2.4, confirms the results that the algorithm achieves minimum variance.

5.2. Minimal phase model

Consider the stable plant

$$Y(z^{-1}) = \frac{0.147z^{-3} - 0.91z^{-2} + z^{-1}}{-0.028z^{-3} + 0.269z^{-2} - 0.98z^{-1} + 1}U(z^{-1}) + \frac{1}{-0.028z^{-3} + 0.269z^{-2} - 0.98z^{-1} + 1}E(z^{-1}) \quad \ldots (52)$$

With a characteristic equation

$$T(z^{-1}) = -0.9z^{-1} + 1$$

Solving the pole-placement problem of equation (5) yields to a minimal order F_0 (z^{-1}) and G_0 (z^{-1})

$$F_0(z^{-1}) = -8.806z^{-2} + 10.247z^{-1} + 1$$

$$G_0(z^{-1}) = -1.677z^{-2} + 9.3z^{-1} - 10.167$$

The minimum variance that the system can attend is calculated from equation (21) and equals to $J_{min} = 1.0$. Measuring the output variance and arranging a similar Table as in Table-1. It is important to highlight that the output variance when using minimal order of *F* and *G* equals to 125.2636.

Table-3. Variance of a zero response with increasing n_P.

n_P	n_ϕ	$\left\lVert \frac{F_0 + q^{-d}BP_{n_P}}{T}C \right\rVert^2_{L_2}$	$\left\lVert \frac{R}{B_-^+} \right\rVert^2_{L_2}$
1	0	10.5518	1.0
2	1	5.9319	1.0
5	4	1.548	1.0
10	9	1.0152	1.0
20	19	1.0	1.0

Table-4. Variance of zero response with the algorithm proposed by Halpern [10].

n_P	n_ϕ	$\left\lVert \frac{F_0 + q^{-d}BP_{n_P}}{T}C \right\rVert^2_{L_2}$	$\left\lVert \frac{R}{B_-^+} \right\rVert^2_{L_2}$
1	0	4.4711	1.0
2	1	4.9705	1.0
5	4	4.3877	1.0
10	9	4.6961	1.0
20	19	4.8875	1.0

Utilizing the algorithm of Halpern [10] yields to the results illustrated in the Table-4. From Table-4, the algorithm of Halpern [10] lacks the consistency in reducing the variance with increasing the order of polynomial *F*. This is explained due to the neglecting of polynomial *T* in setting the constraint minimization.

6. CONCLUSIONS

This work demonstrates a new method to determine the coefficients of a Reduced Variance Controller for a linear time invariant system. The algorithm is applicable for both minimum and non-minimum phase systems. This novel method retains a consistent reduction in the variance.

It has been proven analytically that increasing the order of the coefficients using the prescribed algorithm decreases the variance of the system output towards the minimum variance.

REFERENCES

[1] Karl Johan Astrom and Bjorn Wittenmark 1996. Computer-controlled systems: Theory and Design. Prentice Hall Information and System Sciences Series, 3rd Edition. pp. 460-470.

[2] Zhiyong Chen and Jie Huang. 2009. Attitude tracking and disturbance rejection of rigid spacecraft by adaptive control. IEEE Transactions on Automatic Control. 54(3): 600-605.

[3] D. W. Clarke and P. J. Gawthrop. 1975. A self-tuning controller. IEEE Proceeding. 122(9): 929-934.

[4] D. W. Clarke and P. J. Gawthrop. 1979. Self-tuning controller. IEE Proceeding Part D: Control Theory and Applications. 126(6): 633-640.

[5] R. Davies and M. B. Zarrop. 1998. On reduced variance overparameterized pole assignment control. International Journal of Control. 69(1): 131-144.

[6] Suna Ertunc, Bulent Akay, Havva Boyacioglu and Hale Hapoglu. 2009. Self- tuning control of dissolved oxygen concentration in a batch bioreactor. Food and Bioproducts Processing. 87: 46-55.

[7] D. Fragopoulos, M. J. Grimble and U. Shaked. 1991. H∞ controller design for the siso case using a wiener approach. IEEE Transactions on Automatic Control. 36(10): 1204-1208.

[8] Hong-Wei Gao, Gong-You Tang and Chao Li. 2006. Optimal disturbance rejection with zero steady-state error for time delay systems. In: Proceedings of the 6th World Congress on Intelligent Control and Automation. Dalian, China. June. pp. 511-515

[9] C. Gokcek, P.T. Kabamba and S.M. Meerkov. 2000. Disturbance rejection in control systems with saturating actuators. In: Proceedings of the American Control Conference. pp. 740-744.

[10] M. E. Halpern. 1988. Modified pole-assignment controller for plant models with exact or near pole-zero cancellation. IEE Proceeding. 135(3): 189-195.

[11] A. J. Krener n.d. A brief tutorial on linear and nonlinear control theory. Technical report, Department of Mathematics, University of California.

[12] Ananth Krishnan and Manohar Das. 2000. Minimum variance and LQG control for active noise cancellation-a comparison. In: Proc. 43rd IEEE Midwest Symp. On Circuits and Systems. pp. 1358-1361.

[13] Renato A. Krohling and Joost P. Rey. 2001. Design of optimal disturbance rejection PID controllers using genetic algorithms. IEEE Transactions on evolutionary computation. 5(1): 78-82.

[14] H. Kwakernaak. 1985. Minimax frequency domain performance and robustness optimization of linear feedback systems. IEEE Transactions on Automatic Control. 30(10): 994-1004.

[15] Ioan D Landau. and Gianluca Zito. 2006. Digital Control Systems Design, Identification, and Implementation, chapter Design of Digital Controllers in the Presence of Random Disturbances. Communications and Control Engineering. Springer, London. pp. 169-199.

[16] Anna Patete, Katsuhisa Furuta and Masayoshi Tomizuka. 2006. Self-tuning of repetitive controllers based on generalized minimum variance criterion. In: Proceedings of American Control Conference. pp. 5468-5473.

[17] M. Sternad and A. Ahlen. 1993. Polynomial methods in optimal Control and filtering, chapter LQ control design and self-tuning. Number 49 in IEE Control Engineering. Peter Peregrinus Ltd., United Kingdom.

[18] J. Q. Xie and M. Farooq. 1998. Some issues on disturbance rejection for linear sampled data. In: 24th Annual Conference of the IEEE. pp. 1499-1504.

[19] Toru Yamamoto, Akira Inoue and Sirish L. Shah. 1999. Generalized minimum variance self-tuning pole-assignment controller with PID structure. In: Proceedings of the IEEE International Conference on Control Applications. pp. 125-130.

[20] M. B. Zarrop and M. Fisher. 1985. Reduced-variance pole assignment self- tuning regulation. Int. J. Control. 42: 1013-1033.