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FUZZY TRANSPORTATION PROBLEM WITH ADDITIONAL RESTRICTIONS

Debashis Dutta and A. Satyanarayana Murthy
Department of Mathematics, National Institute of Technology Warangal, A.P, India
E-Mail: dduttamath@gmail.com

ABSTRACT

This paper deals with the transportation problem with additional impurity restrictions where costs are not deterministic numbers but imprecise ones. Here, the elements of the cost matrix are subnormal fuzzy intervals with strictly increasing linear membership functions. By the Max-Min criterion suggested by Bellman and Zadeh [7], the fuzzy transportation problem can be treated as a mixed integer nonlinear programming problem. We show that this problem can be simplified into a linear fractional programming problem. This fractional programming problem is solved by the method given by Kanti Swarup [12].

Keywords: fuzzy numbers, nonlinear programming, fractional programming, impurity constraints.

1. INTRODUCTION

The transportation problem is one of the earliest applications of linear programming problems. The basic transportation problem was originally developed by Hitchcock [5]. The objective of the transportation problem is to determine the optimal amounts of a commodity to be transported from various supply points to various demand points so that the total transportation cost is a minimum. The unit costs i.e. the cost of transporting one unit from a particular supply point to a particular demand point, the amounts available at the supply points and the amounts required at the demand points are the parameters of the transportation problem.

In practice, the parameters of the transportation problem are not always exactly known and stable. This imprecision may follow from the lack of exact information or may be a consequence of a certain flexibility the given enterprise has in planning its capacities. A frequently used means to express the imprecision are the fuzzy numbers.
An algorithm to obtain an integer optimal solution has been presented in [8].

This algorithm requires solving a parametric transportation problem with a parameter in the demand and supply values.

A procedure for solving a fuzzy solid transportation problem was presented by Jimenez and Verdegay [4].

Fuzzy programming and additive fuzzy programming techniques for multi-objective transportation problems were discussed in [1, 2].

A geometric programming approach for a multi-objective transportation problem was considered by Islam and Roy [10].

A method for solving the fuzzy assignment problem was given by Lin and Wen [3].

The multi-objective time transportation problem with additional impurity restriction was studied by Singh and Saxena [6].

In the present paper, the transportation problem with fuzzy costs with strictly increasing linear membership functions and with additional impurity constraints is transformed into a linear fractional programming problem. This problem is solved by the Kanti Swarup's method and computational results show that the proposed method gives an optimal solution to the problem. The optimal shipments from the origins to various destinations are integers, provided the supply and demand values are integers.

2. MATHEMATICAL FORMULATION OF THE PROBLEM

The mathematical formulation of the fuzzy transportation problem with additional restrictions is

$$\text{Min} \sum_{i=1}^{m} \sum_{j=1}^{n} \tilde{c}_{ij} x_{ij}$$

Subject to

$$\sum_{j=1}^{n} x_{ij} = a_i, \ (i=1,2,\ldots,m)$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \ (j=1,2,\ldots,n) \qquad (2.1)$$

$$\sum_{i=1}^{m} f_i x_{ij} \le p_j, \ (j=1,2,\ldots,n)$$

Here a_i is the amount of commodity available at the i^{th} supply point and b_j is the requirement of the commodity at the j^{th} demand point. One unit of the commodity at the i^{th} supply point contains f_i units of impurity. Demand point j cannot receive more than p_j units of impurities and x_{ij} is the amount of commodity transported from the i^{th} supply point to the j^{th} demand point.

The fuzzy costs $\tilde{c}_{ij} = (\alpha_{ij},\beta_{ij})$ (i = 1,2,…,m, j = 1,2,…,n) are subnormal fuzzy numbers having strictly increasing linear membership functions .
The membership function of $\tilde{c}_{ij}$ is

$$\begin{cases} q_{ij} & \text{if } c_{ij} = \beta_{ij}, x_{ij} > 0 \end{cases}$$

$\mu_{ij}(c_{ij}) = \begin{cases} q_{ij}(c_{ij}-\alpha_{ij})/(\beta_{ij}-\alpha_{ij}) & \text{if } \alpha_{ij} \le c_{ij} \le \beta_{ij},\ x_{ij} > 0 \\ 0 & \text{otherwise} \end{cases}$ (2.2)

The condition $x_{ij} > 0$ is added to (2.2) because there is no real expense if $x_{ij} = 0$ in any feasible solution X of (2.1).

We use the notation $< \alpha_{ij}, \beta_{ij} >$ to denote $\tilde{c}_{ij}$. Matrix $\tilde{c}_{ij}$ is shown as follows

$[\tilde{c}_{ij}] = [< \alpha_{ij}, \beta_{ij} >]_{m\times n}$

Matrix $[q_{ij}]$ is defined by $[q_{ij}] = [q_{ij}]_{m\times n}$

Let $\tilde{c}_T$ denote the total cost and the numbers a and b are defined as the lower and upper bounds of the total cost, respectively. We define the membership function of $\tilde{c}_T$ as the linear monotonically decreasing function in (2.3) and use the notation < a, b > to denote fuzzy interval $\tilde{c}_T$. Numbers a and b are constants and subjectively chosen by the manager. We may take a as the minimum cost of the transportation problem with α_{ij}'s as costs and b as the maximum cost of the transportation problem with β_{ij}'s as costs, the demand and supply values in both cases being same as those of problem (2.1).

The membership function of the total cost is

$$\mu_T(c_T) = \mu_T\left(\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}\right) = \begin{cases} 1, & \text{if } c_T \le a \\ \left(b - \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}\right)/(b-a), & \text{if } a \le c_T \le b \\ 0, & c_T \ge b \end{cases} \quad (2.3)$$

3. SOLUTION OF THE PROBLEM

Following the Bellman-Zadeh's criterion [7], we maximize the minimum of the membership functions corresponding to that solution i.e.

$$\text{Max-Min}(\mu_{ij}(i=1,2,\ldots,m,\ j=1,2,\ldots,n), \mu_T(c_T)) \quad (3.1)$$

Where x_{ij} is an element of a feasible solution X of (2.1). Then we can represent the problem as follows

Max-Min $(\mu_{ij}, \mu_T(c_T))$

$x_{ij} > 0$

Subject to

$$\sum_{j=1}^{n} x_{ij} = a_i, \ (i=1,2,\ldots,m)$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \ (j=1,2,\ldots,n) \quad (3.2)$$

$$\sum_{i=1}^{m} f_i x_{ij} \le p_j, \ (j=1,2,\ldots,n)$$

$x_{ij} \ge 0$ for i=1,2,…,m, j=1,2,…,n

By membership functions of (2.2) and (2.3) we can further represent (3.2) as the following equivalent model.

Max λ

Subject to

$\lambda x_{ij} \le q_{ij}(c_{ij}^{\lambda} - \alpha_{ij})x_{ij} / (\beta_{ij} - \alpha_{ij})$ for i=1,2,…,m, j=1,2,…,n

$$\lambda \le \left(b - \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}\right)/(b-a),$$

$$\sum_{j=1}^{n} x_{ij} = a_i, \ (i=1,2,\ldots,m)$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \ (j=1,2,\ldots,n) \quad (3.3)$$

$$\sum_{i=1}^{m} f_i x_{ij} \le p_j, \ (j=1,2,\ldots,n)$$

$c_{ij}^{\lambda} x_{ij} \le \beta_{ij} x_{ij}$ for i=1,2,…,m, j=1,2,…,n

$x_{ij} \ge 0$ for i=1,2,…,m,j=1,2,…,n

Where c_{ij}^{λ} denotes the λ-cut of $\tilde{c}_{ij}$. In (3.3), since x_{ij}, c_{ij}^{λ}, and λ are all decision variables, it can be treated as a mixed integer nonlinear programming model.

We define the set E as the set of all pairs (i,j) where x_{ij} is an element of the feasible solution X of (2.1) and confine our discussion based on E. Then, we can simplify (3.3) as follows

Max λ

Subject to

$\lambda \le q_{ij}(c_{ij}^{\lambda} - \alpha_{ij}) / (\beta_{ij} - \alpha_{ij})$ for $(i,j) \in E$

$$\lambda \le \left(b - \sum_{(i,j) in E} c_{ij}^{\lambda} x_{ij}\right)/(b-a), \quad (3.4)$$

$c_{ij}^{\lambda} \le \beta_{ij}$ for $(i,j) \in E$

We let $d_{ij} = \beta_{ij} - c_{ij}^{\lambda} \ge 0$. Then (3.4) can be expressed as follows

$$\text{Max } \lambda \quad (3.5.0)$$

Subject to

$\lambda \le q_{ij}(\beta_{ij} - \alpha_{ij} - d_{ij}) / (\beta_{ij} - \alpha_{ij})$ for $(i,j) \in E$ (3.5.1)

$\lambda \le (b- \sum_{(i,j) in E} (\beta_{ij} - d_{ij})\, x_{ij}) / (b-a)$, (3.5.2)

$d_{ij}, \lambda \ge 0$ for $(i,j) \in E$ (3.5.3)

Theorem 1

Let λ_x be the optimal value of (3.5.0)-(3.5.3). Suppose

$b < (\sum_{(i,j) in E} (\beta_{ij} - d_{ij})x_{ij} - a \min\{q_{ij}/(i,j) \in E\})/(1 \min\{q_{ij}/(i,j) \in E\})$

Then $\lambda_x = q_{ij} (\beta_{ij} - \alpha_{ij} - d_{ij}) / (\beta_{ij} - \alpha_{ij})$ for $(i,j) \in E$

$= (b- \sum_{(i,j) in E} (\beta_{ij} - d_{ij})\, x_{ij}) / (b-a)$

Proof

The problem (3.5.0) - (3.5.3) can be written into a linear programming model as

Max λ (3.6.0)

Subject to

$d_{ij} + \lambda(\beta_{ij} - \alpha_{ij}) / q_{ij} \le (\beta_{ij} - \alpha_{ij})$ for $(i,j) \in E$ (3.6.1)

$- \sum_{(i,j) in E} d_{ij}x_{ij} + (b-a)\lambda \le b- \sum_{(i,j) in E} \beta_{ij}x_{ij}$ (3.6.2)

$\lambda, d_{ij} \ge 0$ for $(i,j) \in E$

We obtain the dual problem of the above problem as

Min $\sum_{(i,j) in E} (\beta_{ij} - \alpha_{ij})\, w_i + (b- \sum_{(i,j) in E} \beta_{ij}x_{ij})\, w_{m+n}$ (3.7.0)

Subject to

$w_i - w_{m+n}x_{ij} \ge 0$ for$(i,j) \in E$ (3.7.1)

$\sum_{(i,j) in E} ((\beta_{ij} - \alpha_{ij}) / q_{ij})w_i + (b-a)w_{m+n} \ge 1$ (3.7.2)

$w_i \ge 0$ fori=1,2,…,m+n

Let $s_1, s_2, \ldots s_{m+n}$ be the slack variables of (3.6.1) and (3.6.2) respectively. Similarly, let $u_1, u_2, \ldots, u_{m+n}$ be the surplus variables of (3.7.1) and (3.7.2) respectively.

since $b < (\sum_{(i,j) in E} (\beta_{ij} - d_{ij})x_{ij} - a \min\{q_{ij}/ (i,j) \in E\})/(1- \min\{q_{ij}/(i,j) \in E\})$

We have $\min \{q_{ij}/ (i,j) \in E\} > (b- \sum_{(i,j) in E} (\beta_{ij} - d_{ij})x_{ij})/(b-a)$

By (3.5.2) we have $\lambda < \min \{q_{ij}/ (i,j) \in E\}$ and $d_{ij} > 0$ for $(i,j) \in E$.

Based on the complementary slackness theorem, we obtain

$u_1 = u_2 = \ldots = u_{m+n} = 0$

Hence $w_i - w_{m+n}x_{ij} = 0$ for i=1, 2… m+n

Now, if $w_{m+n}= 0$ then $w_1=w_2=\ldots=w_{m+n-1}= 0$.

This is a contradiction to (3.7.2)

Hence $w_{m+n} > 0$

Assuming that the solution is non-degenerate

$w_i = w_{m+n}x_{ij} > 0$

i.e $w_1>0, w_2>0,\ldots,w_{m+n}>0$

Again by complementary slackness theorem

$s_1=s_2=\ldots = s_{m+n} =0$

Therefore $\lambda_x = q_{ij} (\beta_{ij} - \alpha_{ij} - d_{ij}) / (\beta_{ij} - \alpha_{ij})$ for $(i,j) \in E$

$= (b- \sum_{(i,j) in E} (\beta_{ij} - d_{ij})\, x_{ij}) / (b-a)$

In most of the real world problems, the upper bound condition of the total cost $\tilde{c}_T$ i.e

$b < (\sum_{(i,j) in E} (\beta_{ij} - d_{ij})x_{ij} - a \min\{q_{ij}/ (i,j) \in E\})/(1-\min\{q_{ij}/(i,j) \in E\})$

can be just satisfied. Therefore, we concentrate our discussion in this situation.

Theorem 2

Let λ_x be the optimal value of (3.5.0) - (3.5.3) and

$b < (\sum_{(i,j) in E} (\beta_{ij} - d_{ij})x_{ij} - a \min\{q_{ij}/(i,j) \in E\})/(1- \min\{q_{ij}/ (i,j) \in E\})$.

Also let $\gamma_{ij} = (\beta_{ij} - \alpha_{ij})/q_{ij}$ for i=1,2,…,m,j=1,2,…,n.

Then $\lambda_x = (b- \sum_{(i,j) in E} \alpha_{ij}x_{ij}) / (b-a + \sum_{(i,j) in E} \gamma_{ij}x_{ij})$

Proof:

By theorem 1, assuming the solution to be Non-degenerate, we have

$\lambda_x = ((\beta_{ij} - \alpha_{ij} - d_{ij})x_{ij}) / (\gamma_{ij}x_{ij})$ for $(i,j) \in E$

$= (b- \sum_{(i,j) in E} (\beta_{ij} - d_{ij})\, x_{ij}) / (b-a)$

Hence, by componendo and dividendo, we get

$\lambda_x = (b- \sum_{(i,j) in E} (\beta_{ij} - d_{ij})x_{ij} + \sum_{(i,j) in E} (\beta_{ij} - \alpha_{ij} - d_{ij})x_{ij}) / (b-a + \sum_{(i,j) in E} \gamma_{ij}x_{ij})$

$= (b- \sum_{(i,j) in E} \alpha_{ij}x_{ij}) / (b-a + \sum_{(i,j) in E} \gamma_{ij}x_{ij})$ (3.8)

4. THE FRACTIONAL PROGRAMMING MODEL

By Theorem 2 and (3.8), (3.3) can be restated as

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$$\text{Max} \quad \left(b-\sum_{i=1}^{m}\sum_{j=1}^{n} \alpha_{ij}x_{ij}\right) / \left(b-a+\sum_{i=1}^{m}\sum_{j=1}^{n} \gamma_{ij}x_{ij}\right)$$

Subject to

$$\sum_{j=1}^{n} x_{ij} = a_i, \ (i=1,2,\ldots,m)$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \ (j=1,2,\ldots,n) \qquad (4.1)$$

$$\sum_{i=1}^{m} f_i x_{ij} \le p_j, \ (j=1,2,\ldots,n)$$

$x_{ij} \ge 0$ for i=1,2,…,m,j=1,2,…,n

This is a linear fractional programming problem and its optimal solution may be obtained by Kanti Swarup's algotithm [12].
Now d_{ij} for (i,j) ∈ E can be obtained from

$\lambda_x = (\beta_{ij} - \alpha_{ij} - d_{ij})/\gamma_{ij}$ for (i,j) ∈ E

Then the fuzzy costs corresponding to the maximal value of λ are given by
$c_{ij}^{\lambda} = \beta_{ij} - d_{ij}$

5. NUMERICAL EXAMPLE

Consider the problem

$$\text{Minimize} \sum_{i=1}^{3}\sum_{j=1}^{3} \tilde{c}_{ij}x_{ij}$$

Subject to

$$\begin{aligned}
&x_{11}+ x_{12} + x_{13} = 4\\
&x_{21}+ x_{22} + x_{23} = 5\\
&x_{31} + x_{32} + x_{33} = 6\\
&x_{11} + x_{21} + x_{31} = 5\\
&x_{12} + x_{22} + x_{32} = 5\\
&x_{13} + x_{23} + x_{33} = 5\\
&2x_{11} + x_{21} + 0.x_{31} \le 4\\
&2x_{12} + x_{22} + 0.x_{32} \le 1\\
&2x_{13} + x_{23} + 0.x_{33} \le 9\\
&x_{ij} \ge 0 \text{ for } i,j = 1,2,3
\end{aligned} \qquad (5.1)$$

$$\text{Where } \tilde{c}_{ij} = \begin{pmatrix} <4, 13> & <3, 12> & <2, 6> \\ <4, 13> & <6, 14> & <7, 15> \\ <7, 10> & <4, 8> & <6, 12> \end{pmatrix}$$

$$[q_{ij}] = \begin{pmatrix} 0.9 & 0.6 & 0.8 \\ 0.9 & 0.8 & 0.8 \\ 0.6 & 0.8 & 0.6 \end{pmatrix}$$

Then we have

$$[\alpha_{ij}] = \begin{pmatrix} 4 & 3 & 2 \\ 4 & 6 & 7 \\ 7 & 4 & 6 \end{pmatrix} \quad [\beta_{ij}] = \begin{pmatrix} 13 & 12 & 6 \\ 13 & 14 & 15 \\ 10 & 8 & 12 \end{pmatrix} \quad [\gamma_{ij}] = \begin{pmatrix} 10 & 15 & 5 \\ 10 & 10 & 10 \\ 5 & 5 & 10 \end{pmatrix}$$

a is taken as the minimum cost of the transportation problem with costs as α_{ij}'s (a = 54)
b is taken as the maximum cost of the transportation problem with costs as β_{ij}'s (b = 192)
Hence, by (4.1), problem (5.1) can be formulated as

$$\text{Max} \ (192 - 4x_{11} - 3x_{12} - 2x_{13} - 4x_{21} - 6x_{22} - 7x_{23} - 7x_{31} - 4x_{32} - 6x_{33}) \ / (138 + 10x_{11} + 15x_{12} + 5x_{13} + 10x_{21} + 10x_{22} \ 10x_{23} + 5x_{31} + 5x_{32} + 10x_{33})$$

Subject to

$$\begin{aligned}
&x_{11}+ x_{12} + x_{13} = 4\\
&x_{21}+ x_{22} + x_{23} = 5\\
&x_{31} + x_{32} + x_{33} = 6\\
&x_{11} + x_{21} + x_{31} = 5\\
&x_{12} + x_{22} + x_{32} =5\\
&x_{13} + x_{23} + x_{33} =5\\
&2x_{11} + x_{21} + 0.x_{31} \le 4\\
&2x_{12} + x_{22} + 0.x_{32} \le 1\\
&2x_{13} + x_{23} + 0.x_{33} \le 9\\
&x_{ij} \ge 0 \text{ for } i,j = 1,2,3
\end{aligned} \qquad (5.2)$$

The optimal solution of problem (5.2) is obtained by using Kanti Swarup's method as

$x_{13} = 4, x_{21} = 4, x_{23} = 1, x_{31} = 1, x_{32} = 5$ with max $\lambda_x = 0.563$

For (i,j) ∈ E, we have

$\lambda_x = (\beta_{ij} - \alpha_{ij} - d_{ij})/\gamma_{ij}$ so that $d_{ij} = \beta_{ij} - \alpha_{ij} - \lambda_x\gamma_{ij}$

∴ We have

$d_{13} = 4 - (0.563)\ (5) = 1.185$
$d_{21} = 9 - (0.563)\ (10) = 3.37$
$d_{23} = 8 - (0.563)\ (10) = 2.37$
$d_{31} = 3 - (0.563)\ (5) = 0.185$
$d_{32} = 4 - (0.563)\ (5) = 1.185$

The fuzzy costs corresponding to λ = 0.563 are
$c_{ij}^{\lambda} = \beta_{ij} - d_{ij}$ for (i,j) ∈ E
∴ We have
$c_{13}^{0.563} = 6-1.185 = 4.815$
$c_{21}^{0.563} = 13 - 3.37 = 9.63$
$c_{23}^{0.563} = 15 - 2.37 = 12.63$
$c_{31}^{0.563} = 10 - 0.185 = 9.815$
$c_{32}^{0.563} = 8 - 1.185 = 6.815$

$$\therefore \text{Total transportation cost} = \sum_{(i,j) in E} c_{ij}^{0.563}x_{ij} = 114.3$$

6. CONCLUSIONS

In this paper, a fuzzy transportation problem with additional impurity restrictions is formulated into a mixed Nonlinear programming problem by Bellman Zadeh approach. This problem is then framed into a linear fractional programming problem (LFPP). This LFPP is solved using the Kanti Swarup's Method. The optimal degree of satisfaction was obtained as 0.563.The costs at the optimal degree of satisfaction were obtained using known expressions.

In contrast to the classical transportation problem, the existence of a feasible solution of the fuzzy transportation problem considered in this paper is not guaranteed. The non existence of a feasible solution is due to the impurity restrictions. In such cases, feasibility can be attained by increasing the impurity limits of the demand points.

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