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QUINTIC B-SPLINES GALERKIN METHOD FOR FIFTH ORDER BOUNDARY VALUE PROBLEMS

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ABSTRACT

A finite element method involving Galerkin method with quintic B-splines as basis functions has been developed to solve fifth order special case boundary value problems. The basis functions are redefined into a new set of basis functions which vanish on the boundary where the Dirichlet types of boundary conditions are prescribed. The method is tested for solving both linear and non-linear boundary value problems and is compared with the methods available in literature.

Keywords: Galerkin method, quintic B-splines, basis functions, fifth order boundary value problems, absolute error.

1. INTRODUCTION

In the present paper, we have developed a finite element method involving Galerkin method with quintic B-splines as basis functions for the numerical solution of fifth order special case boundary value problems which are in the form

$$y^{(5)}(x) + f(x)y(x) = g(x) \quad a < x < b \tag{1.1}$$

subject to

$$y(a) = a_0, y(b) = b_0, \ y'(a) = a_1, y'(b) = b_1, \ y''(a) = a_2 \tag{1.2}$$

where a_0, a_1, a_2, b_0, b_1 are finite real constants and $f(x)$, $g(x)$ are continuous functions on $[a, b]$. Generally, these types of differential equations arise in the mathematical modeling of viscoelastic fluids [1, 2]. Also we can find the existence and uniqueness of the solution for these types of problems in the book written by Agarwal [3]. From the literature we can observe that there are some authors who worked on these types of boundary value problems by using different methods. Caglar *et al.*, [4] solved fifth order boundary value problems (1) by collocation method with sixth degree B-splines. Ghajala and Shahid [5] solved the boundary value problems of type (1) using sixth degree spline curves. Lamini *et al.*, [6] developed two methods for the solution of boundary value problems of type (1).

In finite element method, the approximate solution can be written as a linear combination of basis functions which constitute a basis for the approximation space under consideration. The finite element method involves methods like Rayleigh-Ritz method, Collocation method, Galerkin method, Petrov-Galerkin method etc. In Galerkin method, a weak form of approximate solution for a given differential equation is existing and is unique under appropriate conditions [7, 8] irrespective of properties of a given differential operator and weak solution is also a classical solution of given differential equation provided sufficient attention is given to boundary conditions [9]. That means the basis functions should vanish on the boundary where the Dirchlet type of boundary conditions are prescribed. Therefore, in this paper we intend to use the Galerkin method with quintic B-splines as basis functions to approximate the solution of given problem.

In section 2 of this paper the definition of quintic B-splines has been described. In section 3, the description of the Galerkin method with quintic B-splines as basis functions has been presented. In section 4, solution procedure to find the nodal parameters has been presented. In section 5, the proposed method is tested on two linear problems and one non-linear problem. The solution of non-linear problem has been obtained as the limit of sequence of linear problems generated by the quasilinearization technique [10]. Finally in the last section the conclusions of the paper are presented.

2. DEFINITION OF QUINTIC B-SPLINES

The cubic B-splines are defined in [11, 12]. The quintic B-splines with evenly spaced knots are defined in [13]. The existence of the quintic spline interpolate $s(x)$ to a function in a closed interval $[a, b]$ for spaced knots (need not be evenly spaced) $a = x_0 < x_1 < x_2 < \Lambda < x_{n-1} < x_n = b$ is established by constructing it. The construction of $s(x)$ is done with the help of the quintic B-splines. Introduce ten additional knots x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4} and x_{n+5} such that

$$x_{-5} < x_{-4} < x_{-3} < x_{-2} < x_{-1} < x_0$$
$$\text{and} \quad x_n < x_{n+1} < x_{n+2} < x_{n+3} < x_{n+4} < x_{n+5}.$$

Now the quintic B-splines $B_i(x)$ are defined by

$$B_i(x) = \sum_{r=i-3}^{i+3} \frac{(x_r - x)_+^5}{\pi'(x_r)}, \quad x \in [x_{i-3}, x_{i+3}]$$
$$= 0 \qquad \text{otherwise}$$

where

$$(x_r - x)_+^5 = (x_r - x)^5, \quad \text{if} \quad x_r \ge x$$
$$= 0 \qquad \text{if} \quad x_r \le x$$

and

$$\pi(x) = \prod_{r=i-3}^{i+3} (x - x_r).$$

Here the set

$$\{B_{-2}(x), B_{-1}(x), B_0(x), \Lambda, B_n(x), B_{n+1}(x), B_{n+2}(x)\}$$

forms a basis for the space $S_5(\pi)$ of fifth degree polynomial splines. The quintic B-splines are the unique non zero splines of smallest compact support with knots at $x_{-5} < x_{-4} < x_{-3} < x_{-2} < x_{-1} < x_0 < \ldots < x_n < x_{n+1} < x_{n+2} < x_{n+3} < x_{n+4} < x_{n+5}$.

3. DESCRIPTION OF THE METHOD

To solve the fifth order special case boundary value problem (1) by Galerkin method with quintic B-splines as basis functions, we define the approximation for $y(x)$ as

$$y(x) = \sum_{j=-2}^{n+2} \alpha_j B_j(x) \tag{2}$$

where α_j's are the nodal parameters to be determined.

In Galerkin method the basis functions should vanish on the boundary where the Dirichlet types of boundary conditions are specified. But, in the set of quintic B-splines $\{B_{-2}(x), B_{-1}(x), B_0(x), B_1(x), \ldots, B_{n-1}(x), B_n(x), B_{n+1}(x), B_{n+2}(x)\}$ the basis functions $B_{-2}(x)$, $B_{-1}(x)$, $B_0(x)$, $B_1(x)$, $B_2(x)$, $B_{n-2}(x)$, $B_{n-1}(x)$, $B_n(x)$, $B_{n+1}(x)$ and $B_{n+2}(x)$ are not vanishing at one of the boundary points. So, there is a necessity of redefining the basis functions into a new set of basis functions which vanish on the boundary where the Dirichlet types of boundary conditions are specified. The procedure for redefining the basis functions is given below.

Using the quintic B-splines described in section 2 and the boundary conditions (1.2), we get the approximate solutions at the boundary points as

$$y(a) = y(x_0) = \sum_{j=-2}^{2} \alpha_j B_j(x_0) = a_0 \tag{3}$$

$$y(b) = y(x_n) = \sum_{j=n-2}^{n+2} \alpha_j B_j(x_n) = b_0 \tag{4}$$

Now eliminating α_{-2}, α_{n+2} from the equations (2), (3) and (4) we get the approximation for $y(x)$ as

$$y(x) = w(x) + \sum_{j=-1}^{n+1} \alpha_j \overline{B}_j(x) \tag{5}$$

where

$$w(x) = \frac{a_0}{B_{-2}(a)} B_{-2}(x) + \frac{b_0}{B_{n+2}(b)} B_{n+2}(x) \tag{6}$$

and

$$\overline{B}_j(x) = \begin{cases} B_j(x) - \dfrac{B_j(x_0)}{B_{-2}(x_0)} B_{-2}(x), & \text{for } j = -1,0,1,2 \\ B_j(x), & \text{for } j = 3,4,\ldots,n-3 \\ B_j(x) - \dfrac{B_j(x_n)}{B_{n+2}(x_n)} B_{n+2}(x), & \text{for } j = n-2, n-1, n, n+1 \end{cases} \tag{7}$$

Here the new set of functions is $\{\overline{B}_j(x)$, $j = -1, 0, \ldots, n, n+1\}$ and they vanish on the boundary. $w(x)$ defined in (6) takes care of Dirichlet type boundary conditions defined in (1.2).

Applying the Galerkin method with the redefined set of basis functions $\overline{B}_j(x)$, $j = -1, 0, 1\ldots n-1, n, n+1$ to the problem (1), we get

$$\int_{x_0}^{x_n} y^{(5)}(x)\overline{B}_i(x)dx + \int_{x_0}^{x_n} f(x)y(x)\overline{B}_i(x)dx = \int_{x_0}^{x_n} g(x)\overline{B}_i(x)\,dx ,$$

for $i = -1, 0\ldots n, n+1$ (8)

Integrating by parts, the first term on the left side of (8) and using the boundary conditions in (1.2), we get

$$\int_{x_0}^{x_n} y^{(5)}(x)\overline{B}_i(x)dx = \int_{x_0}^{x_n} y'(x)\frac{d^4\overline{B}_i}{dx^4}dx - \left.\frac{d\overline{B}_i}{dx}\right|_{x_n} y^{(3)}(x_n) + \left.\frac{d\overline{B}_i}{dx}\right|_{x_0} y^{(3)}(x_0) + \left.\frac{d^2\overline{B}_i}{dx^2}\right|_{x_n} y^{(2)}(x_n) - \left.\frac{d^2\overline{B}_i}{dx^2}\right|_{x_0} a_2 - \left.\frac{d^3\overline{B}_i}{dx^3}\right|_{x_n} b_1 + \left.\frac{d^3\overline{B}_i}{dx^3}\right|_{x_0} a_1 \tag{9}$$

Now substitute (9) in (8) and then substitute the approximation for $y(x)$ given by (5). After rearranging the terms, we will get a system of equations

$$\mathbf{A\alpha} = \mathbf{b} \tag{10}$$

where

$$\mathbf{A} = [a_{ij}];\ a_{ij} = \int_{x_0}^{x_n} \left[\frac{d^4\overline{B}_i}{dx^4}\frac{d\overline{B}_j}{dx} + f(x)\overline{B}_i(x)\overline{B}_j(x) \right] dx - \left.\frac{d\overline{B}_i}{dx}\right|_{x_n} \overline{B}_j^{(3)}(x_n) + \left.\frac{d\overline{B}_i}{dx}\right|_{x_0} \overline{B}_j^{(3)}(x_0) + \left.\frac{d^2\overline{B}_i}{dx^2}\right|_{x_n} \overline{B}_j^{(2)}(x_n) \tag{11}$$

for $i = -1,0,1,\ldots, n, n+1$; $j = -1,0,1,\ldots, n, n+1$.

$$\mathbf{b} = [b_i];\quad b_i = \int_{x_0}^{x_n} \left[g(x)\overline{B}_i(x) - f(x)w(x)\overline{B}_i(x) - \frac{d^4\overline{B}_i}{dx^4} w'(x) \right] dx + \left.\frac{d\overline{B}_i}{dx}\right|_{x_n} w^{(3)}(x_n) - \left.\frac{d\overline{B}_i}{dx}\right|_{x_0} w^{(3)}(x_0) - \left.\frac{d^2\overline{B}_i}{dx^2}\right|_{x_n} w^{(2)}(x_n) + \left.\frac{d^2\overline{B}_i}{dx^2}\right|_{x_0} a_2 + \left.\frac{d^3\overline{B}_i}{dx^3}\right|_{x_n} b_1 - \left.\frac{d^3\overline{B}_i}{dx^3}\right|_{x_0} a_1 \tag{12}$$

for $i = -1, 0, 1,\ldots, n, n+1$.

and

$$\boldsymbol{\alpha} = \left[\alpha_{-1}, \alpha_0, \alpha_1, \alpha_2, \Lambda, \alpha_n, \alpha_{n+1}\right]^T \tag{13}$$

4. SOLUTION PROCEDURE TO FIND THE NODAL PARAMETERS

A typical integral element in any of the matrices **A** and **b** is $\sum_{m=0}^{n-1} \mathrm{I}_m$, where $\mathrm{I}_m = \int_{x_m}^{x_{m+1}} \mathrm{r}_i(x)\mathrm{r}_j(x)\mathrm{Z}(x)dx$ and $\mathrm{r}_i(x)$, $\mathrm{r}_j(x)$ are the basis functions or their derivatives. It may be

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noted that $I_m = 0$ if $(x_{i-3}, x_{i+3}) \cap (x_{j-3}, x_{j+3}) \cap (x_m, x_{m+1}) = \phi$. To evaluate each I_m, we employed 6-point Gauss-Legendre quadrature formula. Thus the stiff matrix **A** is an eleven diagonal band matrix. The nodal parameter vector α has been obtained from the system $\mathbf{A}\alpha = \mathbf{b}$ using a band matrix solution package.

5. NUMERICAL EXAMPLES

To demonstrate the applicability of the proposed method for solving the fifth order special case boundary value problems of type (1), we considered three examples of two linear boundary value problems and one non linear boundary value problem. These examples have been chosen because either analytical or approximate solutions are available for comparison.

Example 1

Consider the following fifth order linear boundary value problem

$$y^{(5)}(x) - y(x) = -(15+10x)\, e^{x}, \quad 0 \le x \le 1 \qquad (14.1)$$

subject to

$$y(0) = y(1) = 0,\ y'(0) = 1,\ y'(1) = -e,\ y^{(2)}(0) = 0 \qquad (14.2)$$

The exact solution for the above system is given by

$$y(x) = x(1 - x)\, e^{x}.$$

We have solved the problem (14) by the proposed method. We have taken the number of intervals for the space variable domain as 10. The approximate solution obtained by the proposed method is compared with the exact solution in Table-1. The maximum absolute error obtained by the proposed method is compared with that of obtained by Caglar *et al*., [4], Ghajala and Shahid [5] in Table-2.

Table-1. Numerical results for Example-1.

x	Solution by the proposed method	Exact solution
0.1	0.0994647	0.0994654
0.2	0.1954230	0.1954244
0.3	0.2834686	0.2834703
0.4	0.3580353	0.3580379
0.5	0.4121780	0.4121803
0.6	0.4373073	0.4373085
0.7	0.4228872	0.4228881
0.8	0.3560854	0.3560865
0.9	0.2213630	0.2213643

Table-2. Maximum absolute error (MAE) $| y(x_i) - y_i |$ for Example 1.

Caglar *et al* [4] with h = 1/10	Ghajala and shahid [5] with h = 1/10	Proposed method with h = 1/10
0.01570	2.2593×10^{-4}	2.6524×10^{-6}

Example 2

Consider the following fifth order linear boundary value problem

$$y^{(5)}(x) + xy(x) = 19x\cos(x) + 2x^3\cos(x) + 41\sin(x) - 2x^2\sin(x), \quad -1 \le x \le 1 \qquad (15.1)$$

subject to $y(-1) = y(1) = \cos(1)$,

$$y'(-1) = -y'(1) = -4\cos(1) + \sin(1),$$
$$y^{(2)}(-1) = 3\cos(1) - 8\sin(1). \qquad (15.2)$$

The exact solution for the above system is given by

$$y(x) = (2x^2 - 1)\cos(x).$$

We have solved the problem (15) by the proposed method. We have taken the number of intervals for the space variable domain as 10. The approximate solution obtained by the proposed method is compared with the exact solution in Table-3. The maximum absolute error obtained by the proposed method is 1.8775×10^{-5}.

Table-3. Numerical results for Example-2.

x	Solution by the proposed method	Exact solution
-0.8	0.1950798	0.1950778
-0.6	-0.2310981	-0.2310939
-0.4	-0.6263329	-0.6263214
-0.2	-0.9016750	-0.9016612
0.0	-1.000016	-1.0000000
0.2	-0.9016800	-0.9016612
0.4	-0.6263367	-0.6263214
0.6	-0.2311043	-0.2310939
0.8	0.1950727	0.1950778

Example 3

Consider the following fifth order non linear boundary value problem

$$y^{(5)}(x) + 24e^{-5y} = \frac{48}{(1+x)^5}, \quad 0 \le x \le 1 \qquad (16.1)$$

subject to

$$y(0) = 0,\ y(1) = \ln(2),\ y'(0) = 1,\ y'(1) = 0.5,\ y^{(2)}(0) = -1 \qquad (16.2)$$

The exact solution for the above system is given by

$$y(x) = \ln(1 + x).$$

Applying the quasilinearization technique [10] to the equation (16.1), we get a sequence of linear problems as

$$y_{r+1}^{(5)}(x) - 120\exp(-5y_r)y_{r+1} = \frac{48}{(1+x)^5} - 120y_r\exp(-5y_r) - 24\exp(-5y_r),$$

$$\text{for } r = 0, 1, 2, \ldots \qquad (17)$$

We have solved the sequence of problems (17) along with boundary conditions (16.2) by the proposed method. We have taken the number of intervals for the space variable domain as 10. The approximation to the solution is converged in two iterations. The approximate solution obtained by the proposed method is compared

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with the exact solution in Table-4. The maximum absolute error obtained by the proposed method is compared with that of obtained by Caglar *et al.*, [4] in Table-5.

Table-4. Numerical results for Example-3.

x	**Solution by the proposed method**	**Exact solution**
0.1	0.0953096	0.0953102
0.2	0.1823205	0.1823216
0.3	0.2623635	0.2623643
0.4	0.3364711	0.3364722
0.5	0.4054652	0.4054651
0.6	0.4700060	0.4700036
0.7	0.5306317	0.5306283
0.8	0.5877893	0.5877867
0.9	0.6418564	0.6418539

Table-5. Maximum absolute error (MAE) $|y(x_i) - y_i|$ for Example 3.

Caglar *et al* [4] with h = 1/30	**Proposed method with h = 1/10**
0.046	3.3974×10^{-6}

6. CONCLUSIONS

In this paper, we have developed a Galerkin method with quintic B-splines as basis functions to solve fifth order special case boundary value problems. The quintic B-spline basis set has been redefined into a new set of basis functions which vanish on the boundary where the Dirichlet boundary conditions are prescribed. The proposed method is applied to solve two linear problems and one non-linear problem to test the efficiency of the method. The numerical results obtained by the proposed method are in good agreement with the exact solutions available in the literature. The maximum absolute errors obtained by the proposed method are less when compared with those of available in the literature. The objective of this paper is to present an accurate and simple method to solve the fifth order special case boundary value problems.

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