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SEPTIC B-SPLINE COLLOCATION METHOD FOR SIXTH ORDER BOUNDARY VALUE PROBLEMS

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ABSTRACT

In this paper sixth order boundary value problems is solved numerically by collocation method. The solution is approximated as a linear combination of septic B-spline functions. The septic B-splines constitute a basis for the space of septic polynomial splines. In the method, the basis functions are redefined into a new set of basis functions which in number match with the number of selected collocation points. To test the efficiency of the method, several numerical examples of sixth order linear and nonlinear boundary value problems are solved by the proposed method. Numerical results obtained by the proposed method are in good agreement with the exact solutions available in the literature.

Keywords: sixth order boundary value problems, collocation method, septic B-splines, band matrix, absolute error.

1. INTRODUCTION

Generally, sixth order boundary value problems arise in several branches of applied mathematics and physics. In the book written by Chandrasekhar [1], we can find that when an infinite horizontal layer of fluid is heated from below and is under the action of rotation, instability sets in. When this instability is an ordinary convection, the ordinary differential equation is a sixth order ordinary differential equation. In this paper, we considered a sixth order boundary value problem of the type

$$y^{(6)} + f(x)\, y(x) = g(x), \qquad a < x < b \tag{1}$$

subject to

$$\begin{aligned} y(a) &= A_0, \quad y(b) = B_0, \\ y'(a) &= A_1, \quad y'(b) = B_1 \\ y''(a) &= A_2, \quad y''(b) = B_2 \end{aligned} \tag{2}$$

Where $f(x)$ and $g(x)$ are continuous functions on $[a, b]$ and A_0, A_1, A_2, B_0, B_1 and B_2 are finite real constants.

The existence and uniqueness of solution of such type of boundary value problems can be found in the book written by Agarwal [2]. El-Gamel *et al.,* [3] used Sinc-Galerkin method to solve sixth order boundary value problems. Akram and Siddiqi [4] solved the boundary value problem of type (1) and (2) with non-polynomial spline technique. Siddiqi *et al.,* [5] solved the same boundary value problems using quintic splines. Also Siddiqi and Akram [6] used septic splines to solve the boundary value problems of type (1) and (2). Lamini *et al.*, [7] used spline collocation method to solve the sixth order boundary value problems. In this paper, we try to present a simple collocation method using septic B-splines as basis functions to solve the sixth order boundary value problem of type (1) and (2).

In finite element method (FEM) the approximate solution can be written as a linear combination of basis functions, which constitute a basis for the approximation space under consideration. FEM involves variational methods like Rayleigh Ritz, Galerkin, Least Squares and Collocation etc., The collocation method seeks an approximate solution by requiring the residual of the differential equation to be identically zero at N selected points (collocation points) in the given space variable domain where N is the number of basis functions in the basis. That means to get an accurate solution by the collocation method one needs a set of basis functions which in number match with the number of collocation points selected in the given space variable domain [8] and also the collocation method is the easiest to implement among the variational methods of FEM. That's why we intend to use the collocation method to solve the sixth order boundary value problem of type (1) and (2).

In section 2 of this paper, the definition of the septic B-splines has been described. In section 3, the description of the method is presented. In section 4, the consistency of the system is discussed. Numerical results obtained by the proposed method are presented in section 5. The proposed method is tested on three linear and two nonlinear sixth order boundary value problems. The solution of a nonlinear boundary value problem is obtained as the limit of sequence of solutions of linear problems generated by the quasilinearization technique [9]. Finally in the last, the conclusions of the paper are presented.

2. DEFINITION OF SEPTIC B-SPLINES

The cubic B-splines and quintic B-splines are defined in [10, 11]. In a similar analogue, the existence of the seventh degree spline interpolate $s(x)$ to a function in a closed interval $[a,b]$ for spaced knots $a=x_0 < x_1 <...< x_{n-1} < x_n=b$ is established by constructing it. The construction of $s(x)$ is done with the help of the septic B-splines.

Introduce fourteen additional knots x_{-7}, x_{-6}, x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}, x_{n+5}, x_{n+6} and x_{n+7} such that

$x_{-7} < x_{-6} < x_{-5} < x_{-4} < x_{-3} < x_{-2} < x_{-1} < x_0$
and $x_n < x_{n+1} < x_{n+2} < x_{n+3} < x_{n+4} < x_{n+5} < x_{n+6} < x_{n+7}$.

Now the seventh degree B-splines $B_i(x)$'s are defined by

$$B_i(x) = \sum_{r=i-4}^{i+4} \frac{(x_r - x)_+^7}{\pi'(x_r)}, \quad x \in [x_{i-4}, x_{i+4}]$$
$$= 0 \qquad \text{otherwise}$$

where

$$(x_r - x)_+^7 = (x_r - x)^7, \text{ if } x_r \ge x$$
$$= 0 \qquad \text{if } x_r \le x$$

and

$$\pi(x) = \prod_{r=i-4}^{i+4} (x - x_r)$$

Here the set $\{B_{-3}(x), B_{-2}(x), B_{-1}(x), B_0(x),..., B_n(x), B_{n+1}(x), B_{n+2}(x), B_{n+3}(x)\}$ forms a basis for the space $S_7(\pi)$ of seventh degree polynomial splines [12]. The septic B-splines are the unique non zero splines of smallest compact support with knots at

$x_{-7} < x_{-6} < x_{-5} < x_{-4} < x_{-3} < x_{-2} < x_{-1} < x_0 < \ldots < x_n < x_{n+1} < x_{n+2} < x_{n+3} < x_{n+4} < x_{n+5} < x_{n+6} < x_{n+7}$.

3. DESCRIPTION OF THE METHOD

To solve the sixth order special case boundary value problem (1) and (2) by the collocation method with septic B-splines as basis functions, we define the approximation for y(x) as

$$y(x) = \sum_{j=-3}^{n+3} \alpha_j B_j(x) \tag{3}$$

where α_i's are the nodal parameters to be determined.

To apply the collocation method one has to select the collocation points in the given space variable domain. These collocation points in number should match with the number of basis functions in the approximation. Here we have taken the mesh points as the selected collocation points. In the approximation (3) we can observe that the number of basis functions is $n + 7$. But the number of mesh points (collocation points) in the space variable domain is n+1. So, there is a necessity to redefine the basis functions into a new set, which should contain $n + 1$ basis functions. For this, we proceed in the following manner.

Using the definition of sixth order B-splines described in section 2 and the boundary conditions (2), we get the approximation for $y(x)$, $y'(x)$ and $y''(x)$ at the boundary points as

$$y(a) = y(x_0) = \sum_{j=-3}^{3} \alpha_j B_j(x_0) = A_0 \tag{4}$$

$$y(b) = y(x_n) = \sum_{j=n-3}^{n+3} \alpha_j B_j(x_n) = B_0 \tag{5}$$

$$y'(a) = y'(x_0) = \sum_{j=-3}^{3} \alpha_j B'_j(x_0) = A_1 \tag{6}$$

$$y'(b) = y'(x_n) = \sum_{j=n-3}^{n+3} \alpha_j B'_j(x_n) = B_1 \tag{7}$$

$$y''(a) = y''(x_0) = \sum_{j=-3}^{3} \alpha_j B''_j(x_0) = A_2 \tag{8}$$

$$y''(b) = y''(x_n) = \sum_{j=n-3}^{n+3} \alpha_j B''_j(x_n) = B_2 \tag{9}$$

Eliminating α_{-3}, α_{-2}, α_{-1}, α_{n+1}, α_{n+2} and α_{n+3} from the equations (3) to (9), we get the approximation for $y(x)$ as

$$y(x) = w(x) + \sum_{j=0}^{n} \alpha_j \widetilde{B}_j(x) \tag{10}$$

where

$$w(x) = w_2(x) + \frac{Q_{-1}(x)}{Q''_{-1}(x_0)}\left(A_2 - w''_2(x_0)\right)$$
$$+ \frac{Q_{n+1}(x)}{Q''_{n+1}(x_n)}\left(B_2 - w''_2(x_n)\right) \tag{11}$$

$$w_2(x) = w_1(x) + \frac{\left(A_1 - w'_1(x_0)\right)}{P'_{-2}(x_0)} P_{-2}(x)$$
$$+ \frac{\left(B_1 - w'_1(x_n)\right)}{P'_{n+2}(x_n)} P_{n+2}(x) \tag{12}$$

$$w_1(x) = \frac{B_{-3}(x)}{B_{-3}(x_0)} A_0 + \frac{B_{n+3}(x)}{B_{n+3}(x_n)} B_0 \tag{13}$$

$$\widetilde{B}_j(x) = \begin{cases} Q_j(x) - \dfrac{Q_{-1}(x)}{Q''_{-1}(x_0)} Q''_j(x_0) & j = 0,1,2,3 \\ Q_j(x) & j = 4,5,...,n-4 \\ Q_j(x) - \dfrac{Q_{n+1}(x)}{Q''_{n+1}(x_n)} Q''_j(x_n) & j = n-3, n-2, n-1, n \end{cases} \tag{14}$$

$$Q_j(x) = \begin{cases} P_j(x) - \dfrac{P_{-2}(x)}{P'_{-2}(x_0)} P'_j(x_0) & j = 0,1,2,3 \\ P_j(x) & j = 4,5,...,n-4 \\ P_j(x) - \dfrac{P_{n+2}(x)}{P'_{n+2}(x_n)} P'_j(x_n) & j = n-3, n-2, n-1, n \end{cases} \tag{15}$$

$$P_j(x) = \begin{cases} B_j(x) - \dfrac{B_{-3}(x)}{B_{-3}(x_0)} B_j(x_0) & j = 0,1,2,3 \\ B_j(x) & j = 4,5,...,n-4 \\ B_j(x) - \dfrac{B_{n+3}(x)}{B_{n+3}(x_n)} B_j(x_n) & j = n-3, n-2, n-1, n \end{cases} \tag{16}$$

Now the new set of basis functions is

$\{\widetilde{B}_j(x), j = 0, 1, 2,\ldots, n\}$

and the number of basis functions match with the number of selected collocation points.

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Applying the collocation method with the redefined set of basis functions $\widetilde{B}_j(x)$, j = 0, 1, 2, ..., n to the problem (1), we get

$$\left\{ \left.\frac{d^6 w}{dx^6}\right|_{x_i} + \sum_{j=0}^{n} \alpha_j \left.\frac{d^6 \widetilde{B}_j}{dx^6}\right|_{x_i} \right\} + f(x_i)\left\{ w(x_i) + \sum_{j=0}^{n} \alpha_j \widetilde{B}_j(x_i) \right\} = g(x_i)$$

for $i = 0,1,2,\ldots,n$ (17)

Rewriting the above system of equations in the matrix form, we get

$A\alpha = b$ (18)

where

$A = [a_{ij}]$; $a_{ij} = \left.\frac{d^6 \widetilde{B}_j}{dx^6}\right|_{x_i} + f(x_i)\widetilde{B}_j(x_i)$,

for $i = 0,1,2,\ldots,n$ and $j = 0,1,2,\ldots,n$ (19)

$b = [b_i]$; $b_i = g(x_i) - \left.\frac{d^6 w}{dx^6}\right|_{x_i} - f(x_i)w(x_i)$,

for $i = 0,1,2,\ldots,n$ (20)

and $\alpha = [\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_n]^T$ (21)

4. CONSISTENCY OF THE SYSTEM

Theorem. *The matrix* **A** *in the system* (18) *is nonsingular.*

Proof. The basis function $\widetilde{B}_j(x)$ is defined only in the interval $[x_{j-4}, x_{j+4}]$ and outside of this interval it is zero. Also at the end points of the interval $[x_{j-4}, x_{j+4}]$ the basis function $\widetilde{B}_j(x)$ vanishes. Therefore, $\widetilde{B}_j(x)$ is having non-vanishing values at the mesh points x_{j-3}, x_{j-2}, x_{j-1}, x_j, x_{j+1}, x_{j+2}, x_{j+3} and at other mesh points the value of $\widetilde{B}_j(x)$ is zero. It is clear that from the definition of septic B-splines defined section 2, the value of $\widetilde{B}_j(x)$ at x_i is dominating for $j = i$ when compared with the values of $\widetilde{B}_j(x_i)$ for $j \neq i$. The derivatives of $\widetilde{B}_j(x)$ upto sixth order also have the same nature at the mesh points as in the case of $\widetilde{B}_j(x)$. Using these facts, we can say that the matrix **A** defined in (19) is a seven diagonal band matrix with nonzero entries and dominant principal diagonal elements. Hence the matrix is nonsingular. □

Since the matrix A is nonsingular, the system (18) is consistent and thus the nodal parameters can be obtained by using band matrix solution package. We have used FORTRAN 77 programming to develop the package and hence obtained the results by the proposed method.

5. NUMERICAL EXAMPLES

To demonstrate the applicability of the proposed method for solving the sixth order special case boundary value problems of type (1) and (2), we considered three linear and two non linear boundary value problems of such type. These examples have been chosen because either analytical or approximate solutions are available in the literature and the solutions obtained by the proposed method are compared with the exact solutions.

Example 1

Consider the following sixth order linear boundary value problem

$y^{(6)} + xy = -(24 + 11x + x^3)$ ex $\quad 0 < x < 1$

subject to $\quad y(0) = y(1) = 0$

$y'(0) = 1,\ y'(1) = -e$ (22)

$y''(0) = 0,\ y''(1) = -4e$

The analytical solution of the above problem is $y(x) = x(1-x)e^x$.

Absolute errors obtained by the proposed method are presented in Table-1. The maximum absolute error obtained by the proposed method for this problem is 2.858×10^{-5}.

Table-1. Numerical results for the example 1 with step length $h = 0.1$.

x	Exact solution	Absolute error by proposed method
0.1	9.946539E-02	5.014241E-06
0.2	1.954244E-01	1.330674E-05
0.3	2.834704E-01	1.162291E-06
0.4	3.580379E-01	1.490116E-07
0.5	4.121803E-01	8.106232E-06
0.6	4.373085E-01	2.160668E-05
0.7	4.228881E-01	2.643466E-05
0.8	3.560865E-01	2.858043E-05
0.9	2.213642E-01	2.457201E-05

Example 2

Consider the following sixth order linear boundary value problem

$y^{(6)} + y = 6\cos x \quad 0 < x < 1$

subject to $\quad y(0) = y(1) = 0$

$y'(0) = -1,\ y'(1) = \sin(1)$ (23)

$y''(0) = 2,\ y''(1) = 2\cos(1)$

The analytical solution of the above problem is $y(x) = (x-1)\sin x$.

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Absolute errors obtained by the proposed method are presented in Table-2. The maximum absolute error obtained by the proposed method for this problem is 1.041×10^{-5}.

Table-2. Numerical results for the example 2 with step length $h = 0.1$.

x	Exact solution	Absolute error by proposed method
0.1	-8.985008E-02	3.695488E-06
0.0	-1.589355E-01	1.041591E-05
0.3	-2.068641E-01	5.930662E-06
0.4	-2.336510E-01	1.007318E-05
0.5	-2.397128E-01	9.059906E-06
0.6	-2.258570E-01	4.082918E-06
0.7	-1.932653E-01	1.043081E-07
0.8	-1.434712E-01	3.576279E-06
0.9	-7.833266E-02	4.805624E-06

Example 3

Consider the following sixth order linear boundary value problem

$$y^{(6)} - y = -6e^{x} \qquad 0 < x < 1$$

subject to

$$y(0) = 1, \quad y(1) = 0$$
$$y'(0) = 0, \quad y'(1) = -e \tag{24}$$
$$y''(0) = -1, \quad y''(1) = -2e$$

The analytical solution of the above problem is $y(x) = (1- x)\, e^{x}$.

Absolute errors obtained by the proposed method are presented in Table-3. The maximum absolute error obtained by the proposed method for this problem is 2.741×10^{-5}.

Table-3. Numerical results for the example 3 with step length $h = 0.1$.

x	Exact solution	Absolute error by proposed method
0.1	9.946538E-01	1.215935E-05
0.2	9.771222E-01	2.741814E-05
0.3	9.449012E-01	2.205372E-06
0.4	8.950948E-01	5.483627E-06
0.5	8.243606E-01	2.503395E-06
0.6	7.288475E-01	1.621246E-05
0.7	6.041259E-01	2.068281E-05
0.8	4.451082E-01	2.261996E-05
0.9	2.459602E-01	1.946092E-05

Example 4

Consider the following sixth order nonlinear boundary value problem

$$y^{(6)} = e^{-x}y^{2}(x) \qquad 0 < x < 1$$

subject to

$$y(0) = 1, \quad y(1) = e$$
$$y'(0) = 1, \quad y'(1) = e \tag{25}$$
$$y''(0) = 1, \quad y''(1) = e$$

The analytical solution of the above problem is $y(x) = e^{x}$.

Applying the quasilinearization technique [9] to the above nonlinear problem (25), we get a sequence of linear problems as

$$y^{(6)}_{n+1} + [-2e^{-x}y_n(x)]y_{n+1} = -e^{-x}\,[y_n^{2}(x)]$$

for $n = 0, 1, 2, \ldots$ (26)

Here y_{n+1} represents $(n + 1)^{th}$ approximation to $y(x)$. The solution of nonlinear problem (25) is obtained by the limit of sequence of solutions of the linear problems (26). Absolute errors obtained by the proposed method are presented in Table-4. The maximum absolute error obtained by the proposed method for this problem is 1.797×10^{-4}.

Table-4. Numerical results for the example 4 with step length h = 0.1.

x	Exact solution	Absolute error by proposed method
0.1	1.105171	2.360344E-05
0.2	1.221403	7.891655E-05
0.3	1.349859	7.212162E-05
0.4	1.491825	1.453161E-04
0.5	1.648721	1.792908E-04
0.6	1.822119	1.491308E-04
0.7	2.013753	1.797676E-04
0.8	2.225541	1.180172E-04
0.9	2.459603	7.367134E-05

Example 5

Consider the following sixth order nonlinear boundary value problem

$$y^{(6)} = e^{x}y^{2}(x) \qquad 0 < x < 1$$

subject to

$$y(0) = 1, \quad y(1) = -1/e$$
$$y'(0) = -1, \quad y'(1) = 1/e \tag{27}$$
$$y''(0) = 1, \quad y''(1) = 1/e$$

The analytical solution of the above problem is $y(x) = e^{-x}$.

Applying the quasilinearization technique [8] to the above nonlinear problem (27), we get a sequence of linear problems as

$$y^{(6)}_{n+1} + [-2e^{x}y_n(x)]y_{n+1} = -e^{x}\,[y_n^{2}(x)]$$

for $n = 0, 1, 2, \ldots$ (28)

Here y_{n+1} represent $(n + 1)^{Th}$ approximation to $y(x)$. The solution of nonlinear problem (27) is obtained by the limit of sequence of solutions of the linear problems (28). Absolute errors obtained by the proposed method are presented in Table-4. The maximum absolute error obtained by the proposed method for this problem is 1.704×10^{-5}.

Table-5. Numerical results for the example 5 with step length h = 0.1.

x	Exact solution	Absolute error by proposed method
0.1	9.048374E-01	7.092953E-06
0.2	8.187308E-01	1.704693E-05
0.3	7.408182E-01	2.443790E-06
0.4	6.703200E-01	1.323223E-05
0.5	6.065307E-01	1.633167E-05
0.6	5.488116E-01	1.013279E-05
0.7	4.965853E-01	1.528859E-05
0.8	4.493290E-01	8.761883E-06
0.9	4.065697E-01	5.245209E-06

6. CONCLUSIONS

In this paper, we have developed a collocation method with septic B-splines as basis functions to solve a sixth order special case boundary value problem. In the collocation method, we have selected the mesh points as collocation points. The septic B-spline basis set has been redefined into a new set in which the number of basis functions is equal to the number of collocation points. The proposed method is applied to solve three linear problems and two non-linear problems to test the efficiency of the proposed method. The numerical results obtained by the proposed method are in good agreement with the exact solutions available in the literature. The objective of this paper is to present a simple method to solve a sixth order special case boundary value problem.

REFERENCES

[1] S. Chandrasekhar. 1961. Hydrodynamic and Hydromagnetic stability. Clarendon Press, Oxford, (Reprinted: Dover Books, New York. 1981).

[2] R.P. Agarwal 1986. Boundary value problems for higher-order differential equations. World Scientific, Singapore.

[3] Mohamed El-Gamel, John R. Cannon and Ahmed I zayed. 2004. Sinc-Galerkin method for solving linear sixth order boundary value problems. Mathematics of computation. 73(247): 1325-1343.

[4] Ghazala Akram, Shahid S. Siddiqi. 2006. Solution of sixth order boundary value problem using non-polynomial spline technique. Applied Mathematics and Computation. 181: 708-720.

[5] Shahid S. Siddiqi, Ghazala Akram and Saima Nazeer. 2007. Quintic spline solution of linear sixth order boundary value problems. Applied Mathematics and Computation. 189: 887-892.

[6] Shahid S. Siddiqi and Ghazala Akram. 2008. Septic spline solutions of sixth order boundary value problems. Journal of computational and applied Mathematics. 215: 288-301.

[7] Abdellah Lamini, Hamid Mraoui, Driss Sbibih, Ahmed Tijini and Ahmed Zidna. 2008. Spline collocation method for solving linear sixth order boundary-value problems. International Journal of Computer Mathematics. 85(11): 1673-1684.

[8] J. N. Reddy. 2005. An introduction to the finite element method. 3rd Edition. Tata Mcgraw Hill Publishing Company ltd., New Delhi.

[9] R.E Bellman and R.E. Kalaba. 1965. Quasilinearization and Nonlinear Boundary value problems. American Elsevier, New York.

[10] C. de Boor. 1978. A Practical Guide to Spline, Springer-Verlag.

[11] I.J. Schoenberg. 1966. On Spline Functions. MRC Report 625, University of Wisconsin.

[12] P. M. Prenter. 1989. Splines and Variational Methods. John-Wiley and Sons, New York.