VOL. 5, NO. 8, AUGUST 2010 ISSN 1819-6608
ARPN Journal of Engineering and Applied Sciences

www.arpnjournals.com

A THIRD ORDER EULER METHOD FOR NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT

There has been a great deal of interest to improve on Euler methods for solving Initial value Problems (IVPs) in Ordinary Differential Equations, because of its easy implementation and low computational cost. In this work, we continue in the spirit of Euler, to develop a new third order Euler Method. The new scheme was implemented on some standard IVPs. Our computational results show that the method is consistent, accurate and convergent of order 3. Succinct overviews of its basic properties necessary for the selection of a good numerical method in the development of program codes are also presented.

Keywords*:* third order Euler method, numerical solutions, differential equations, stability, convergence, initial value problems.

1. INTRODUCTION

Ordinary Differential Equations (ODEs) are one of the important and widely used techniques in mathematical modeling. However, not many ODEs have an analytic solution and even if there is one, usually it is extremely difficult to obtain and it is not very practical. This leads to the need of numerical integration of the Initial Value Problem (IVP) for ODE.

Unfortunately all methods are affected by the presence of different kind of errors at the integration point. In many critical applications, like the motion of a NEO or space missions using a close fly-by to a massive body, the effect of the errors can be critical.

Hence, it is natural to ask for methods providing solutions which are as correct as possible and, better, to guarantee that the solution at a given time is inside a given subset of the phase space. Furthermore this subset should be small, that is, the method should not overestimate its size [2].

Many authors have improved on the popular method of Euler developed between 1768 and 1770 [4] because of its ease of implementation. In this work, we present a new improvement that has led to a third order algorithm, yet maintaining the simple nature of the Euler method.

2. THEORETICAL BACKGROUND

A first-order ODE is of the form:
$y'(x) = f(x, y)$.
The general form of an Initial Value Problems (IVPs) associated with this model is:

$$y'(x) = f(x, y), \;\; y(x_0) = \eta \qquad (1)$$

where $x \in \mathbb{R}, y, \eta \in \mathbb{R}^n$ and $f: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$
Whenever f does not depend explicitly on x, equation (1) is referred to as autonomous system. An approximate solution to an IVP (1) is typically obtained by iterating a set of *difference equations* that approximate the original system. The famous method of Euler was published in his three volume work *Institutiones Calculi Integrals* in the years 1768 to 1770, republished in his collected works in 1913 [5]. It involves computing a discrete set $\{y_n\}$, for arguments $\{x_n\}$, using the difference equation

$$y_{n+1} - y_n = hf(x_n, y_n), \;\; n = 1, 2, \ldots m \qquad (2)$$

Where the step size $h = x_{n+1} - x_n$.

The Euler method is the simplest, not only of all one-step methods, but of all methods for the approximate solution of IVPs. It uses only one piece of information from the past and evaluates the driving function only once per step. However, it is not practical for computational purposes since a considerable effort is required to improve accuracy. In spite of its limitations, the Euler method remains the fundamental building block for the higher accuracy methods, be it Runge–Kutta or Linear Multistep methods [9]. Since the difference equation is linear in y_n and f_n, and being a one-step method, it can easily handle IVPs that require variable steplength. Since Euler proposed his historical Euler method in 1768, there has been lot of developments on this class of method. Runge [10], observed that Euler's method (2) gives rise to a rather inefficient approximation of the integral by the area of a rectangle of height $f(x_0)$.

Thus, he says, "it is already much better" to extend the Midpoint rule and the Trapezoidal rule to differential equations by inserting for the missing y-values the results of Euler steps yielding the following methods:

$$y_{n+1} - y_n = hf\left(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}hf\,(x_n, y_n)\right) \qquad (3)$$

$$y_{n+1} - y_n = \tfrac{1}{2}h\left(f(x_n, y_n) + f(x_n + h, y_n + hfxn, yn \qquad (4)$$

Method (3) is referred to as the *Modified Euler (ME)* or the *Improved Polygon method*, while method (4) is known as the *Improved Euler (IE) method*. In [1], Abraham improved on the Modified Euler by inserting the forward Euler method, in place of y_n in the inner function evaluation of the Modified Euler method.

This improvement led to a new method called Improved Modified Euler (IME) Method . It is given as,

$y_{n+1} - y_n = hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hf\left(x_n + h, y_n + hfxn, yn\right.\right.$ (5)

That is, y_n in ME method (3) was replaced with $y_n + hf(x_n, y_n)$.

However, it was found out that the IME method performed very poorly in comparison with the ME method, with respect to autonomous IVP. Thus, Abraham [1], further proposed a new improvement on Euler Method, which is called Modified Improved Modified Euler Method. This was achieved by using $y_n + \frac{1}{2}hf(x_n, y_n)$ to replace y_n in IME method (5) to develop,

$y_{n+1} - y_n = hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hf\left(x_n + \frac{1}{2}h, y_n + 12hfxn, yn\right.\right.$ (6)

known as *Modified Improved Modified Euler (MIME) method*. The summary of the properties of these methods [1, 8] including the new Third Order Euler Method (TOEM) being proposed are presented in Table 1.

Table 1: Development of Euler Methods

Method	$y_{n+1} - y_n = \Phi_{E\,method}(x_n, y_n; h)$	Stability Function $R_{Method}(z)$
EM	$= hf\ (x_n, y_n)$	$1 + z$
ME	$= hf\ \left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hf\ (x_n, y_n)\right)$	$1 + z + \frac{1}{2}z^2$
IE	$= \frac{1}{2}h\left(f(x_n, y_n) + f\left(x_n + h, y_n + hf(x_n, y_n)\right)\right)$	$1 + z + \frac{1}{2}z^2 + \frac{1}{4}z^3$
IME	$= hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hf\left(x_n + h, y_n + hf(x_n, y_n)\right)\right)$	$1 + z + \frac{1}{2}z^2 + \frac{1}{4}z^3$
MIME	$= hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hf(x_n, y_n)\right)\right)$	$1 + z + \frac{1}{2}z^2 + \frac{1}{4}z^3$
TOEM	$= hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{3}hf(x_n, y_n)\right)\right)$	$1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3$

In this work, we present a modification to MIME method that is capable of yielding a third order accuracy for Euler methods.

3. DEVELOPMENT AND ANALYSIS OF THE NEW THIRD ORDER EULER METHOD

Euler methods, like other one-step methods are based on the principle of discretization. These methods have the common feature that no attempt is made to approximate the exact solution $y(x)$ over a continuous range of the independent variable. Approximate values are sought only on a set of discrete points $x_0, x_1, x_2, \ldots$

We continue in the spirit of Euler and other authors that have improved on Euler method by substituting for y_n in IME method (5), $y_n + \frac{1}{3}hf(x_n, y_n)$ to develop,

$y_{n+1} - y_n = hf(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hf(x_n + \frac{1}{2}h, y_n + \frac{1}{3}hf(x_n, y_n)$ (7)

called *Third Order Euler Method (TOEM).*

The properties of the increment function Φ_{TOEM}, the righthand side of *TOEM* (7) of the newly proposed *TOEM* is in general, very crucial to its stability and convergence characteristics. These properties are also investigated to be able to ascertain how efficient is the new method being proposed. For any standard IVP of an ODE given by (1), we are interested in finding the solution of $y(x)$ by the *TOEM* (7). Whenever the function f does not depend on x the equation (1) is said to be autonomous.

Theorem 3.0.1. *The existence of such a solution $y(x)$ is guaranteed and unique provided that $f(x,y)$:*

- *is continuous in the infinite strip* $\Psi = \{x_0 \le x \le T, |y| < \infty\}$,
- *and is, more specifically, Lipschitz continuous in the dependent variable y over the same region Ψ, i.e., $\exists$ a positive constant L such that*

$\forall (x, y), (x, \hat{y}) \in \Psi$,

$|f(x, y) - f(x, \hat{y})| \le L|y - \hat{y}|$

Actually, these (*sufficient)* conditions also guarantee that the solution depends Lipschitz continuously on the initial condition. i.e., if $y(x)$ is the solution to the original problem and now $\hat{y}(x)$ also satisfies the ODE but with a different initial condition $\hat{y}(x_0)$, then $\exists$ a positive constant K such that

$|y(x) - \hat{y}(x)| \le K|y(x_0) - \hat{y}(x_0)|$

These conditions together with the existence and uniqueness of a solution defines a *well-posed* problem [3]. The following lemma will be useful for establishing the properties.

Lemma 3.0.2. *Let{ δ_i i= 0(1) n} be a set of real numbers. If there exist finite constants* Γ *and* $\prod$ *such that*

$$|\delta_{i+1}| \leq \Gamma|e_i| + \Pi, i = 0\,(1)n-1, \quad (8)$$

Then

$$|\delta_i| \leq \frac{\mathrm{r}^i-1}{\mathrm{r}-1}\prod + \mathrm{r}^i\,|e_0|, \mathrm{r} \neq 1. \quad (9)$$

Proof. When $i = 0$, (9) is satisfied identically as $|e_0| \leq |e_0|$.

Suppose (9) holds whenever $i \leq j$ so that

$$|\delta_j| \leq \frac{\mathrm{r}^j-1}{\mathrm{r}-1}\prod + \mathrm{r}^j\,|e_0|. \quad (10)$$

Then, from (8) $i = j$ implies that

$$|\delta_{j+1}| \leq \mathrm{r}|e_j| + \prod. \quad (11)$$

On substituting (10) into (11), we have

$$|\delta_{j+1}| \leq \frac{\mathrm{r}^{j+1}-1}{\mathrm{r}-1}\prod + \mathrm{r}^{j+1}\,|e_0|. \quad (12)$$

Hence, (9) holds for all $i \geq 0$.

3.1 Stability

The following theorem guarantees the stability of the newly proposed *TOEM* (7).

Theorem 3.1.1. *Suppose the IVP (1) satisfies the hypotheses of theorem (3.0.1), then the TOEM method is stable.*

Proof. Let y_n and z_n be two sets of solutions generated recursively by the *TOEM* with the initial condition

$$y(x_0) = y_0,\ z(x_0) = z_0,\ |y_0 - z_0| = \delta_0$$

Let

$$\delta_n = y_n - z_n,\quad n \geq 0, \quad (13)$$

and

$$y_{n+1} = y_n + h\,\Phi_{TOEM}(x_n, y_n; h), \quad (14)$$

$$z_{n+1} = z_n + h\,\Phi_{TOEM}(x_n, z_n; h). \quad (15)$$

This implies that

$$y_{n+1} - z_{n+1} = y_n - z_n + h\,\{\Phi_{TOEM}(x_n, y_n; h) - \Phi_{TOEM}(x_n, z_n; h)\}. \quad (16)$$

Using (13) and triangle inequality, we have:

$$|\delta_{n+1}| = (1 + hL)\,|\delta_n|, n \geq 0 \quad (17)$$

If we assume $\Gamma = 1 + hL$, and $\Pi = 0$, then Lemma 3.0.2 implies that

$$|\delta_n| \leq K|\delta_0|, \quad (18)$$

where

$$K = e^{l(b-a)} < \propto$$

which implies the stability of the newly proposed TOEM.

3.2 Absolute Stability

Absolute stability analysis of one-step methods is usually carried out using the linear model problem

$$y' = \lambda y, y(x_0) = y_0, \quad x_0 \leq x, \quad (19)$$

where λ is complex. This has the analytical solution

$$y(x) = \eta e^{\lambda(x-x_0)} \quad (20)$$

The problem has a stable fixed point at $y = 0$ for $Re(\lambda) < 0$. The region of absolute stability for a method is then the set of values of h (real and non-negative) and (complex) for which $Y_{n\to 0}$ as $n \to \infty$, that is, for which the fixed point at the origin is stable. Thus, we want the set of values of h and λ for which $|R(h\lambda)| \leq 1$ where R, the stability function, is the eigen value of the Jacobian of the Euler methods map evaluated at the fixed point.

Using the linear model problem (19), the third order Euler method (7) gives the following recurrence relation

$$y_{n+1} = y_n + h\lambda + \left(y_n + \frac{1}{2}h\lambda(y_n + h\lambda y_n)\right) \quad (21)$$

$$= \left(1 + h\lambda + \frac{1}{2}h^2\lambda^2 + \frac{1}{6}h^3\lambda^3\right) y_n \quad (22)$$

The solution which satisfies the initial condition $y_0 = 1$ is

$$y_n = \left(1 + h\lambda + \frac{1}{2}h^2\lambda^2 + \frac{1}{6}h^3\lambda^3\right) \quad (23)$$

In order to examine the convergence of this at, say, x_n , it is necessary to study the behaviour of this function as h tends to zero in such a manner that x_n remains fixed.

Now

$$y_n = \left(1 + h\lambda + \frac{1}{2}h^2\lambda^2 + \frac{1}{6}h^3\lambda^3\right)^{\frac{x_n}{h}} \quad (24)$$

so that

$$\ln y_n = \frac{x_n}{h}\ln\left(1 + h\lambda + \frac{1}{2}h^2\lambda^2 + \frac{1}{6}h^3\lambda^3\right) \quad (25)$$

Then by de l'Hopital's rule

$$\lim_{h\to 0}\frac{1}{h}\ln\left(1 + h\lambda + \frac{1}{2}h^2\lambda^2 + \frac{1}{4}h^3\lambda^3\right)$$

$$= \lim_{h\to 0}\left(\frac{\lambda + h\lambda^2 + \frac{1}{2}h^2\lambda^3}{1 + h\lambda + \frac{1}{2}h^2\lambda^2 + \frac{1}{6}h^3\lambda^3}\right) \quad (26)$$

$$= \lambda \quad (27)$$

Hence,

VOL. 5, NO. 8, AUGUST 2010 ISSN 1819-6608
ARPN Journal of Engineering and Applied Sciences

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$$\lim_{h \to 0} \ln y_n = \lambda x_n \qquad (28)$$

and thus,

$$\lim_{h \to 0} y_n = e^{\lambda x_n} . \qquad (29)$$

Then the method is consistent and convergent to $O(h^4)$. If we let z = hλ then, the general form of the stability function of *MIME method* is:

$$R(z) = 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 \qquad (30)$$

The stability function of the new method shows that it is a third-order method. Its region of absolute stability is also derived from $|R(z)| < 1$, as $-2.51275 < z < 0$

3.3 Convergence

Lemma 3.3.1. Suppose the IVP (1) satisfies the hypothesis of the Existence and Uniqueness Theorem (3.0.1), then the increment function Φ_{MIME} specified by (6) satisfies a Lipschitz condition of order 2 with respect to the independent variable y.

Proof. Suppose L is the Lipschitz constant for $f(x, y)$ w.r.t. y, then, by theorem (3.0.1),

$$|f(x, y) - f(x, \hat{y})| \leq L|y - \hat{y}|.$$

Using (6),

$$|\boldsymbol{\Phi_{TOEM}(x_n, y_n; h) - \Phi_{TOEM}(x_n, z_n; h)}|$$

$$= hf(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hf(x_n + \frac{1}{2}h, y_n + \frac{1}{3}hf(x_n, y_n)$$

$$- hf(x_n + \frac{1}{2}h, z_n + \frac{1}{2}hf(x_n + \frac{1}{2}h, z_n + \frac{1}{3}hf(x_n, z_n)$$

$$< L|y_n - z_n|\{h + \frac{1}{2}h^2 + \frac{1}{6}h^3\} < L^*|y_n - z_n|,$$

where the Lipschitz constant is L^* given as

$$L^* = L\{h + \frac{1}{2}h^2 + \frac{1}{6}h^3\}. \qquad (31)$$

Thus, the proposed method is convergent to $O(h^4)$ and it's order of accuracy is 3.

4. NUMERICAL COMPUTATION

In this section we compute the absolute errors of numerical values of $y(x)$ for the IVPs in Examples 1– 4 [6, 7], shown in table 2, using the Euler methods outlined in the table 1. These computations were carried out using varying stepsizes of h = 0.0005, 0.001, 0.002, 0.004, 0.008, 0.016, 0.032 respectively. The charts of the absolute errors of the numerical values of $y(x)$ are shown in the figures 1-7

Table 2: Initial Value Problems for Implementing the Euler Methods

Example No	$y'(x)$	Initial Conditions	Interval of Integration	Theoretical Solution $y(x)$
1	$-10(y(x)^2 - 1)$	$y(0) = 2$	$0 \leq x \leq 1$	$1 + \frac{1}{10x + 1}$
2	$y(x)$	$y(0) = 1$	$0 \leq x \leq 1$	$exp(x)$
3	$\sqrt{y(x)}$	$y(0) = 1$	$0 \leq x \leq 1$	$\frac{1}{4}(x+2)^2$
4	$1 + (y(x))^2$	$y(0) = 1$	$0 \leq x \leq \frac{\pi}{4}$	$\tan(x + \frac{x}{4})$

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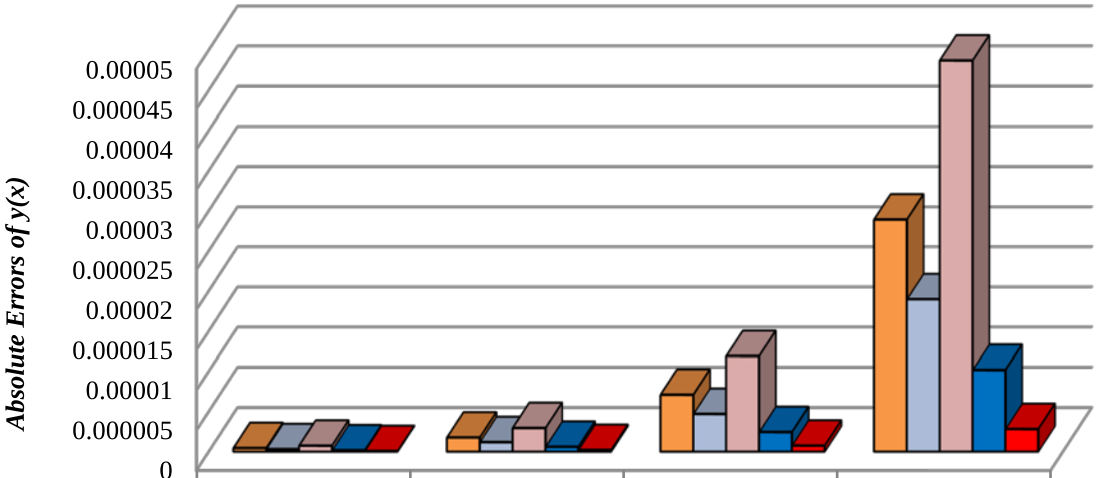
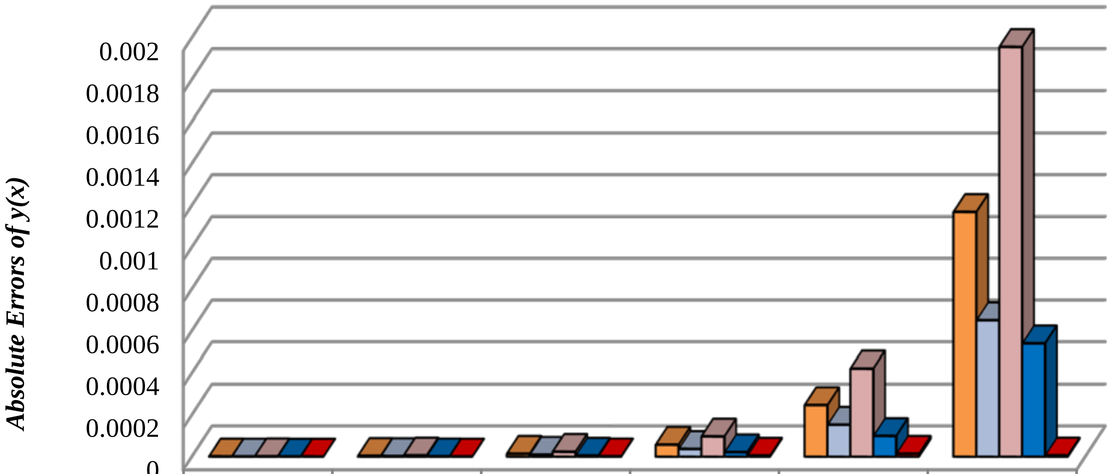


	0.5	0.5	0.5	0.5
ME	4.36147E-07	1.75314E-06	7.08185E-06	2.88965E-05
IE	2.90196E-07	1.16416E-06	4.68372E-06	1.89516E-05
IME	7.28025E-07	2.93071E-06	1.1872E-05	4.86845E-05
MIME	1.46371E-07	5.92326E-07	2.42476E-06	1.01543E-05
TOEM	4.77054E-08	1.88708E-07	7.37509E-07	2.8047E-06

Figure–1. Chart of Absolute Error of Numerical Values of y(x) at x = 0.5 for example 1.



	0.8	0.8	0.8	0.8	0.8	0.8
ME	8.30731E-07	3.35419E-06	1.36733E-05	5.68511E-05	0.000246453	0.001168486
IE	5.51747E-07	2.2192E-06	8.97454E-06	3.66767E-05	0.000152739	0.000650171
IME	1.38854E-06	5.62152E-06	2.3027E-05	9.64541E-05	0.00042053	0.001955515
MIME	2.80499E-07	1.14702E-06	4.79354E-06	2.09019E-05	9.88023E-05	0.000539434
TOEM	8.95565E-08	3.5041E-07	1.33609E-06	4.76944E-06	1.35982E-05	5.57465E-06

Figure–2. Chart of Absolute Error of Numerical Values of y(x) at x = 0.8 for example 1.

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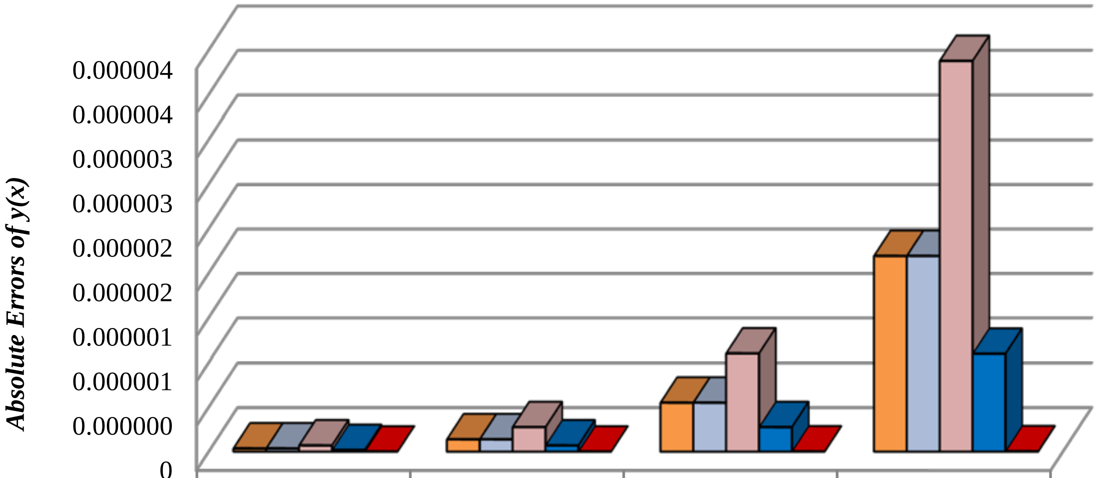
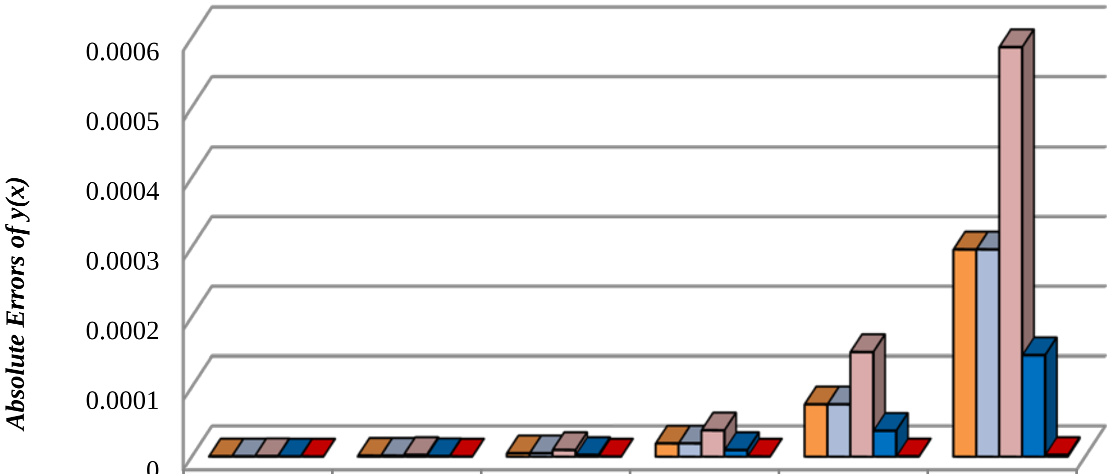


	0.5	0.5	0.5	0.5
ME	3.43355E-08	1.3729E-07	5.4875E-07	2.19171E-06
IE	3.43355E-08	1.3729E-07	5.4875E-07	2.19171E-06
IME	6.86581E-08	2.74478E-07	1.09668E-06	4.37685E-06
MIME	1.71613E-08	6.85937E-08	2.73964E-07	1.09257E-06
TOEM	4.29012E-12	3.43217E-11	2.7435E-10	2.19127E-09

Figure–3. Chart of Absolute Error of Numerical Values of y(x) at x = 0.5 for example 2.



	0.8	0.8	0.8	0.8	0.8	0.8
ME	2.96516E-07	1.18518E-06	4.73359E-06	1.88776E-05	7.50581E-05	0.000296641
IE	2.96516E-07	1.18518E-06	4.73359E-06	1.88776E-05	7.50581E-05	0.000296641
IME	5.9281E-07	2.36858E-06	9.45302E-06	3.76425E-05	0.000149224	0.000586286
MIME	1.48147E-07	5.91699E-07	2.3597E-06	9.38226E-06	3.708E-05	0.000144781
TOEM	7.41296E-11	5.92531E-10	4.73265E-09	3.77403E-08	2.99997E-07	2.36947E-06

Figure–4. Chart of Absolute Error of Numerical Values of y(x) at x = 0.8 for example 2.

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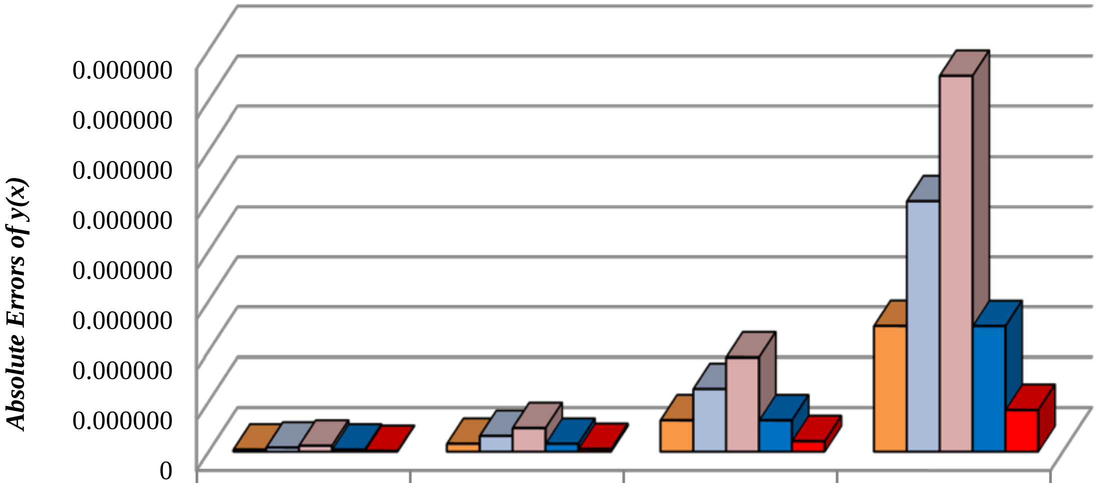
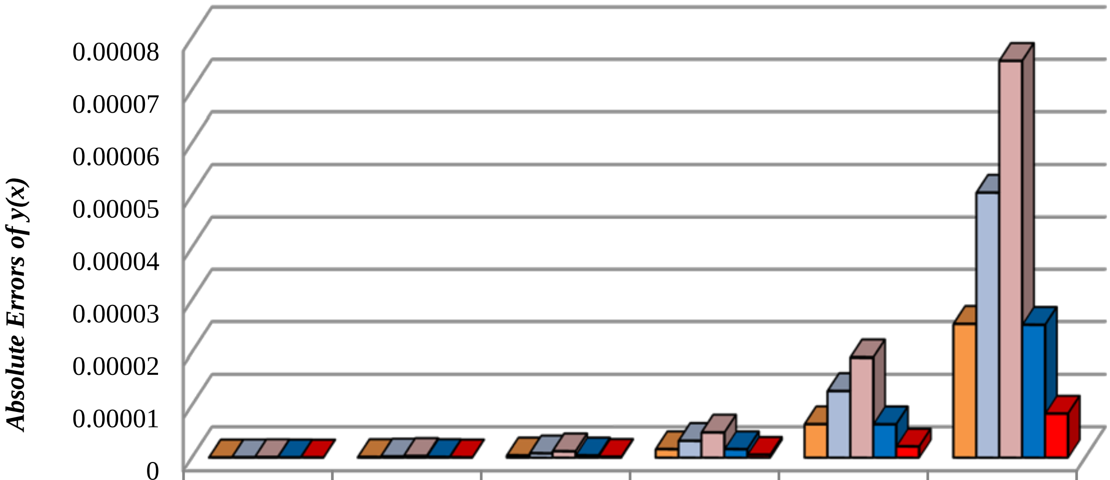


	0.5	0.5	0.5	0.5
ME	3.90581E-09	1.56215E-08	6.24719E-08	2.49775E-07
IE	7.81074E-09	3.12359E-08	1.24888E-07	4.99101E-07
IME	1.17157E-08	4.68504E-08	1.87303E-07	7.48426E-07
MIME	3.90537E-09	1.5618E-08	6.24438E-08	2.4955E-07
TOEM	1.30174E-09	5.2056E-09	2.08115E-08	8.31584E-08

Figure–5. Chart of Absolute Error of Numerical Values of y(x) at x = 0.5 for example 3.



	0.8	0.8	0.8	0.8	0.8	0.8
ME	2.49946E-08	9.99572E-08	3.99657E-07	1.59726E-06	6.37808E-06	2.54249E-05
IE	4.99786E-08	1.99829E-07	7.9863E-07	3.18906E-06	1.27127E-05	5.05054E-05
IME	7.49625E-08	2.997E-07	1.1976E-06	4.78082E-06	1.90468E-05	7.55773E-05
MIME	2.49893E-08	9.99143E-08	3.99314E-07	1.59452E-06	6.35617E-06	2.52497E-05
TOEM	8.32917E-09	3.33E-08	1.33067E-07	5.31201E-07	2.11629E-06	8.39712E-06

Figure–6. Chart of Absolute Error of Numerical Values of y(x) at x = 0.8 for example 3.

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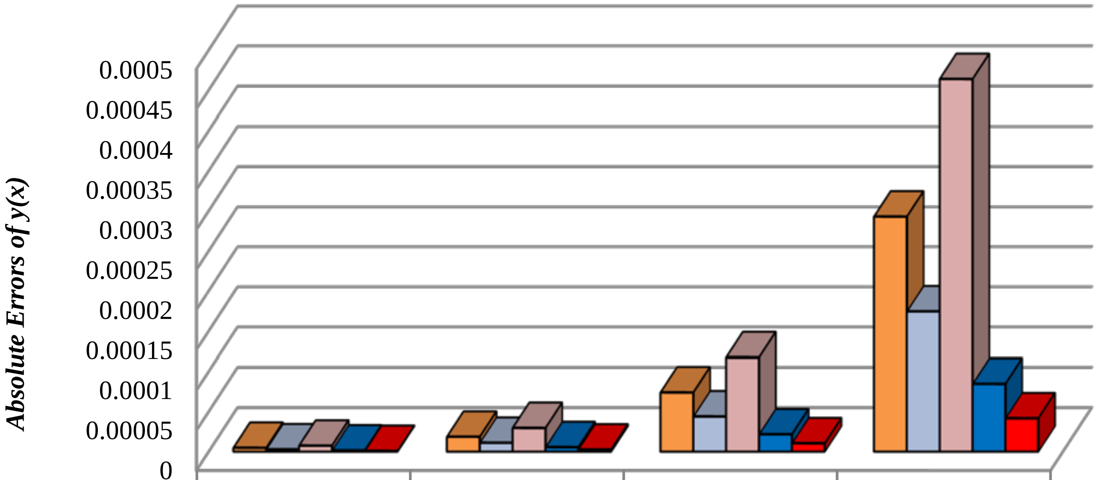

	0.5	0.5	0.5	0.5
ME	4.6363E-06	1.85091E-05	7.37475E-05	0.000292674
IE	2.74369E-06	1.09637E-05	4.3766E-05	0.000174341
IME	7.37313E-06	2.94185E-05	0.000117086	0.000463714
MIME	1.36631E-06	5.43773E-06	2.1532E-05	8.43966E-05
TOEM	6.35027E-07	2.54832E-06	1.02583E-05	4.15432E-05

Figure–7. Chart of Absolute Error of Numerical Values of y(x) at x = 0.5 for example 4.

5. CONCLUSIONS

The comparison between the Third Order Euler Method and other existing Euler methods shows that, TOEM possesses wider region of absolute stability. From the analysis and the computational results displayed in the figures, it is evident that the method is efficient, stable and convergent. The test of convergence and stability function also reveals that the method is of order 3

From the numerical experiments, the proposed method competes well with the popular existing methods. In fact, the absolute errors generated by the method are the least in all our examples.

We therefore conclude that the Third Order Euler Method proposed is efficient and accurate. Higher orders of accuracy of 3, wider region of absolute stability and faster convergence have been achieved over and above the existing Euler Methods.

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