FLUID FLOW IN A RIGID WAVY NON-UNIFORM TUBE: APPLICATION TO FLOW IN RENAL TUBULES

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ABSTRACT

The problem of steady, viscous, incompressible fluid flow in a tube of slowly varying cross-section with absorbing wall is studied. For the mathematical formulation of the problem the effect of fluid absorption through permeable wall is accounted by prescribing flux as a function of axial distance and the fluid is considered as a Newtonian fluid. The nonlinear equations of motion are linearized by perturbation method by assuming δ (ratio of inlet radius to wavelength) as a small parameter and the resulting equations are solved by numerical methods. The effects of reabsorption coefficient (α), slope parameter (k) and amplitude ratio (ε) on the velocity profiles mean pressure drop and wall shear stress are presented graphically. Results indicate that the variation of slope parameter and reabsorption coefficient influences the flow field considerably.

Keywords: fluid flow, tube, renal tubule.

INTRODUCTION

One of the most useful and important organ of a human body is kidney, which excrete end products of body metabolism and controls concentrations of most of the constitutes of body fluids. Each kidney encloses over a million tiny units called Nephrons, which are the basic functional unit of kidney. Nephrons consist of glomerulus and renal tubules which are originating from the tuft of the glomerulus. Renal tubules are involved in one of the most important and final stage of the nephron function, in clearing the end products of metabolism and in maintaining the volume of the body fluids. The major portion of the tubular function is being carried out by the proximal renal tubule, which is highly permeable to water and small solutes, to facilitate their reabsorption from glomerular filtrate. The proximal renal tubules are not uniform all along their length. It is therefore suitable to consider a mathematical model for renal flow with non-uniform tube of varying cross-section with reabsorption at the wall.

Flow in renal tubule has been studied by various authors. Macey [1] was the first to study the mathematical modeling of the flow in proximal renal tubule. He formulated the problem as the flow of an incompressible viscous fluid through a circular tube with linear rate of reabsorption at the wall. Kelman [2] noted that the bulk flow in the proximal tubule decays exponentially with the axial distance. Later, Macey [3] used this condition and solved the equations of motion to find average pressure drop. Marshall and Trowbridge [4] and Palatt *et al* [5] used physical conditions existing at the permeable wall instead of prescribing the flux/radial velocity at the wall.

In all the above analysis the renal tubule is assumed as cylindrical tube of uniform cross-section, while in general such tubes may not have uniform cross-section throughout their length. Radhakrishnamacharya *et al*. [6] made an attempt to understand the flow through the renal tubule by studying the hydrodynamical aspects of an incompressible viscous fluid in a circular tube of varying cross-section with reabsorption at the wall. Chandra and Prasad [7] analyzed flow in rigid tubes of slowly varying cross-section with absorbing wall. Chaturani and Ranganatha [8] considered fluid flow through a diverging/converging tube with variable wall permeability.

In this study we have made an attempt to understand the flow through renal tubule by studying the hydrodynamical aspect of an incompressible viscous fluid in a rigid converging/diverging tube of varying cross-section with reabsorption at the wall. The boundary of the tube wall varies with x. It is taken as

$$\eta(x) = d + k_1 x + a \sin\left(\frac{2\pi x}{\lambda}\right) \tag{1}$$

where d is the radius of the tube at the inlet (at $x = 0$), k_1 is a constant whose magnitude depends on the length of the tube exit and inlet dimensions and which is assumed as << 1 , a is the amplitude and λ is the wave length (Figure-1).

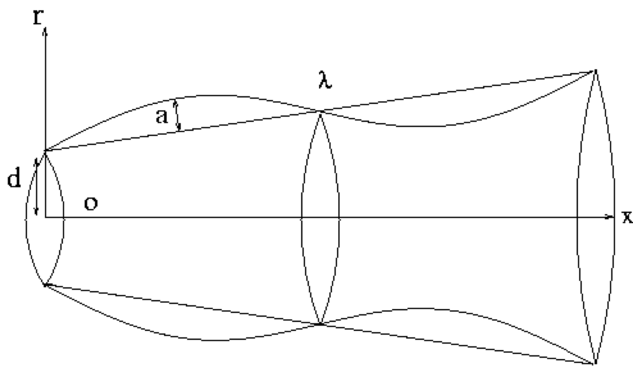

Figure-1. Geometry of 3 dimensional renal tubules.

MATHEMATICAL FORMULATION

Consider an incompressible fluid flows through a tube with slowly varying cross-section as given by equation (1). The motion of the fluid is assumed to be laminar, steady and symmetric. The tube is long enough to neglect the initial and end effects. The governing equations of such fluid motion are given by

$$\frac{\partial u}{\partial x} + \frac{1}{r}\frac{\partial (rv)}{\partial r} = 0 \tag{2}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) \tag{3}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{v}{r}\right)\right) \tag{4}$$

where u and v are the velocity components along the axial and radial axes respectively, p is the pressure, ρ density of the fluid and $\nu = \frac{\mu}{\rho}$ is kinematic viscosity.

The boundary conditions are taken as follows: The tangential velocity at the wall is zero. That is,

$$u + \frac{d\eta}{dx} v = 0 \quad \text{at} \quad r = \eta(x) \tag{5}$$

The regularity condition requires

$$v = 0, \quad \frac{\partial u}{\partial r} = 0 \quad \text{at} \quad r = 0 \tag{6}$$

The reabsorption at the wall has been accounted for by considering the bulk flow as a function of axial distance x, which is decreasing with x. That is, the flux across a cross-section is given as

$$Q = \int_0^{\eta(x)} 2\pi r u(x,r)dr = Q_0 \cdot F(\alpha x) \tag{7}$$

Where $F(\alpha x) = 1$ when $\alpha = 0$ and decreases with x, $\alpha \geq 0$ is the reabsorption coefficient and is a constant, and Q_0 is the flux across the cross-section at $x = 0$.

The relation between stream function ψ and velocity components is given as

$$u = \frac{-1}{r}\frac{\partial \psi}{\partial r}, \qquad v = \frac{1}{r}\frac{\partial \psi}{\partial x} \tag{8}$$

Using the following non-dimensional quantities

$$x' = \frac{x}{\lambda}, \qquad r' = \frac{r}{d}, \qquad \eta' = \frac{\eta}{d},$$

$$\psi' = \frac{2\pi\psi}{Q_0}, \qquad \alpha' = \alpha\lambda, \qquad p' = \frac{2\pi d^3}{\mu Q_0} p$$

and equation (8), the equations (2), (3) and (4) are transformed to the non-dimensional form as (after dropping the primes) :

$$\left(\delta^2 \frac{\partial^2}{\partial x^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2}\right)^2 \psi = \frac{\delta R}{2}\left[-\frac{1}{r}\frac{\partial \psi}{\partial r}\left(\delta^2 \frac{\partial^2}{\partial x^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2}\right)\frac{\partial \psi}{\partial x} + \frac{1}{r}\frac{\partial \psi}{\partial x}\left(\delta^2 \frac{\partial^2}{\partial x^2} - \frac{1}{r}\frac{\partial}{\partial r} \cdots \right.\right. \tag{9}$$

Where, $\delta = \frac{d}{\lambda}$ and $R = \frac{Q_0}{\pi d \nu}$.

Further the boundary conditions (5), (6) and (7) become

$$\frac{\partial \psi}{\partial r} = \frac{\delta(k_1 + A\cos(2\pi x))\partial \psi}{\partial x}$$

$$\text{at} \quad r = \eta(x) = 1 + kx + \varepsilon\sin(2\pi x) \tag{10}$$

$$\psi = \frac{\partial \psi}{\partial x} = 0 \quad \text{at} \quad r = 0 \tag{11}$$

$$\psi = -F(\alpha x)$$

$$\text{at} \quad r = \eta(x) = 1 + kx + \varepsilon\sin(2\pi x) \tag{12}$$

Where $A = \frac{2\pi a}{\lambda}$, $\varepsilon = \frac{a}{d}$, $k = \frac{k_1\lambda}{d}$

The parameter R is the Reynolds number and δ is the wave-number (the ratio of inlet width to the wavelength). ε is amplitude ratio (the ratio of amplitude to the inlet width) and k is slope parameter. In this problem, we consider exponentially decaying bulk flow [3]. That is, in equation (7), F is taken as

$$F(\alpha x) = e^{-\alpha x} \tag{13}$$

ANALYSIS

It may be noted that the flow is quite complex because of nonlinearity of governing equation and the boundary conditions. Thus to solve equation (9) for velocity components, in the present analysis we shall seek a solution for stream function $\psi(x,r)$ in the form of a power series in terms of δ (assuming δ as small parameter), as

$$\psi(x,r) = \psi_0(x,r) + \delta\, \psi_1(x,r) + \ldots \tag{14}$$

Substituting equation (14) in equations (9), (10), (11) and (12) and collecting coefficients of various like powers of δ, we get the following sets of equations for, $\psi_0(x,r)$, $\psi_1(x,r)$, … .

δ^0 Case:

$$\nabla^2 \psi_0 = 0 \tag{15}$$

where $\nabla = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r}$.

The boundary conditions are:

$$\frac{\partial \psi_0}{\partial r} = 0 \quad \text{at} \quad r = \eta(x) \tag{16}$$

$$\psi_0 = \frac{\partial \psi_0}{\partial x} = 0 \quad \text{at} \quad r = 0 \tag{17}$$

$$\psi_0 = -F(\alpha x) = -e^{-\alpha x} \quad \text{at} \quad r = \eta(x) \tag{18}$$

δ^1 Case:

$$\nabla^2\psi_1 = \frac{R}{2}\left[\frac{1}{r}\frac{\partial\psi_0}{\partial x}\nabla\frac{\partial\psi_0}{\partial r} - \frac{1}{r}\frac{\partial\psi_0}{\partial r}\nabla\frac{\partial\psi_0}{\partial x} - \frac{2}{r^2}\frac{\partial\psi_0}{\partial x}\nabla\psi_0 - \frac{1}{r^3}\frac{\partial\psi_0}{\partial r}\frac{\partial\psi_0}{\partial x}\right] \tag{19}$$

where $\nabla = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r}$.

The boundary conditions are:

$$\frac{\partial\psi_1}{\partial r} = \frac{(k_1 + A\cos(2\pi x))(\partial\psi_0)}{\partial x} \quad \text{at} \quad r = \eta(x) \tag{20}$$

$$\psi_1 = \frac{\partial\psi_1}{\partial x} = 0 \quad \text{at} \quad r = 0 \tag{21}$$

$$\psi_1 = 0 \quad \text{at} \quad r = \eta(x) \tag{22}$$

Similar expressions can be written for higher order of δ cases. However, since we are looking for an approximate analytical solution for the problem, we consider up to order of δ^1 equations.

The solution of equation (15) together with equations (16) to (18) is

$$\psi_0(x,r) = A_1 r^4 + A_2 r^2 \tag{23}$$

where $A_1 = \frac{1}{\eta^4}e^{-\alpha x}$ and $A_2 = \frac{-2}{\eta^2}e^{-\alpha x}$.

The solution of equation (19) together with equations (20) to (22) is

$$\psi_1(x,r) = A_3 r^8 + A_4 r^6 + A_5 r^4 + A_6 r^2 \tag{24}$$

where

$$A_3 = \frac{-1}{72} R\, A_1 \frac{dA_1}{dx}$$

$$A_4 = \frac{1}{24} R\, A_2 \frac{dA_1}{dx}$$

$$A_5 = \frac{1}{2}(k_1 + A\cos(2\pi x))\left(\frac{dA_1}{dx}\eta + \frac{dA_2}{dx}\frac{1}{\eta}\right) + R\left(\frac{1}{24}A_1\frac{dA_1}{dx}\eta^4 + \frac{1}{12}A_2\frac{dA_1}{dx}\eta^2\right)$$

$$A_6 = \frac{-1}{2}(k_1 + A\cos(2\pi x))\left(\frac{dA_1}{dx}\eta^3 + \frac{dA_2}{dx}\eta\right) - R\left(\frac{1}{36}A_1\frac{dA_1}{dx}\eta^6 + \frac{1}{24}A_2\frac{dA_1}{dx}\eta^4\right)$$

Hence, substituting ψ_0 and ψ_1 in equation (14), we get that

$$\psi = A_1 r^4 + A_2 r^2 + \delta(A_3 r^8 + A_4 r^6 + A_5 r^4 + A_6 r^2) \tag{25}$$

Now, the non-dimensional pressure $p(x,r)$ can be obtained by using equations (3), (8) and (25). It is given as

$$p(x,r) = \delta\frac{\partial u}{\partial r} + \frac{1}{\delta}\int\frac{\partial^2 u}{\partial r^2}dx + \frac{1}{\delta}\int\frac{1}{r}\frac{\partial u}{\partial r}dx - \frac{R}{2}\left(\int u\frac{\partial u}{\partial x}dx + \int v\frac{\partial u}{\partial r}dx\right) \tag{26}$$

The mean pressure is given as

$$\bar{p}(x) = \frac{1}{\pi\,\eta^2(x)}\int_0^{\eta(x)} 2\,\pi\, r\, p(x,r)\, dr \tag{27}$$

Further, the mean pressure drop between $x = 0$ and $x = x_0$ is

$$\Delta\bar{p}(x_0) = \bar{p}(0) - \bar{p}(x_0) \tag{28}$$

The wall shear stress $\tau_w(x)$ is defined as

$$\tau_w = \frac{(\sigma_{rr} - \sigma_{xx})\frac{d\eta}{dx} + \sigma_{xr}\left(1 - \left(\frac{d\eta}{dx}\right)^2\right)}{1 + \left(\frac{d\eta}{dx}\right)^2} \quad \text{at} \quad y = \eta(x) \tag{29}$$

where $\sigma_{xx} = 2\mu\frac{\partial u}{\partial x}$, $\sigma_{rr} = 2\mu\frac{\partial v}{\partial r}$, $\sigma_{xr} = \mu\left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial x}\right)$.

Using the non-dimensional quantity $\tau_w{}' = \frac{2\,\pi\, d^3}{\mu Q_0}\tau_w$, the wall shear $\tau_w(x)$ becomes (after dropping the primes),

$$\tau_w = \frac{2\,\delta^2\left(\frac{\partial v}{\partial r} - \frac{\partial u}{\partial x}\right)\frac{d\eta}{dx} + \left(\frac{\partial u}{\partial r} + \delta^2\frac{\partial v}{\partial x}\right)\left(1 - \delta^2\left(\frac{d\eta}{dx}\right)^2\right)}{1 + \delta^2\left(\frac{d\eta}{dx}\right)^2} \tag{30}$$

It may be noted that in equation (26), the integrations are difficult to evaluate analytically to get closed form expression for $p(x,r)$. Therefore, they are calculated by numerical integration.

RESULTS AND DISCUSSIONS

The objective of this analysis is to study the behavior of an incompressible fluid flow through a tube of converging/diverging and slowly varying cross-section with absorbing wall. It may be recalled that k characterize the slope of the converging/diverging wavy wall. $k = 0$ represents a rigid channel of slowly varying cross-section (sinusoidal channel). ε and α represents amplitude and reabsorption coefficient of wavy wall.

We discuss the effects of these parameters on the radial velocity ($v(x,r)$), mean pressure drop (Δp) and wall shear stress (τ_w) quantities. In all our numerical calculations, the following parameters are fixed as $A = 0.0628$ and $\delta = 0.1$. We take $R = 1.0$ to consider low Reynolds number flow.

The velocity v :

The radial velocity profile of the flow is obtained by taking different values of k at different cross-sections of the tube $x = 0.1$, 0.5 and 0.75 . The values of k are taken as -0.1 for convergent tube, 0.0 for normal tube (sinusoidal tube) and 0.1 for divergent tube.

It can be observed from Figures 2(a) - 2(c), that the radial velocity is less for the convergent tube and more for the divergent tube than the normal tube case. However, in the downstream of the flow, though there is no significant change in the behavior of radial velocity, the quantity of the velocity decreases.

Figures 3(a) - 3(c) show the effects of $\varepsilon = 0.2$ on the radial velocity at $x = 0.1$, 0.5 and 0.75 . Though the overall qualitative behavior is similar to the case of $\varepsilon = 0$, we observe a significant change in the qualitative nature of v at $x = 0.5$. It may be noted that at $x = 0.5$, wavy nature of wall is narrowing symmetrically and this brings negative values of radial velocity near center of the tube.

Figure-2(a) is compared with Figure-1 of Radhakrishnamacharya *et al* [6] paper. They studied the radial velocity at the entrance while we presented our results for the entire length of tube. So, we compare both results at the entrance $x = 0.1$ for the limiting case $\varepsilon = 0$. It can be observed from these two Figures that at $x = 0.1$, the distribution of velocity with radial coordinates (r), is the same.

Mean pressure drop $\overline{\Delta p}$:

The values of the mean pressure drop (equation (28)) over the length of the tube are calculated for different values of k , ε and α . It can be observed from Figures 4(a) and 4(b) that the mean pressure drop is more for the convergent tube than the normal tube and it is less for divergent tube. It may also be noted that as the values of ε changes from 0.0 to 0.2 , the values of the mean pressure drop changes significantly. Further, as the reabsorption coefficient α increases the mean pressure drop for converging/normal/diverging tubes decreases (Figures 5(a) - 5(c)).

Magnitude of wall shear stress $|\tau_\omega|$:

The effects of k , ε and α on the magnitude of wall shear stress τ_w are studied and presented graphically in Figures 6(a) - 7(c). It may be remarked from Figures 6(a) and 6(b) that the magnitude of wall shear stress is more for the convergent tube and less for the divergent tube than the normal tube. It can also be noted that as the values of ε changes from 0.0 to 0.2 , the values of the magnitude of wall shear stress changes considerably. Moreover, as the reabsorption coefficient α increases, the magnitude of wall shear stress for converging/normal/diverging tubes decreases (Figures 7(a) to 7(c)).

CONCLUSIONS

The main contribution of this study is to see the effect of wavy nature of walls on the flow of incompressible fluid in a rigid tube of slowly varying converging/diverging walls with possible applications to the flow in renal tubules. It is observed that the radial velocity is less for the convergent tube and more for the divergent tube than the normal tube. The mean pressure and the magnitude of the wall shear stress are more for the convergent tube and less for the divergent tube than the normal tube. Moreover, as the reabsorption coefficient α increases, the magnitude of wall shear stress and the mean pressure drop for converging/normal/diverging tubes decreases.

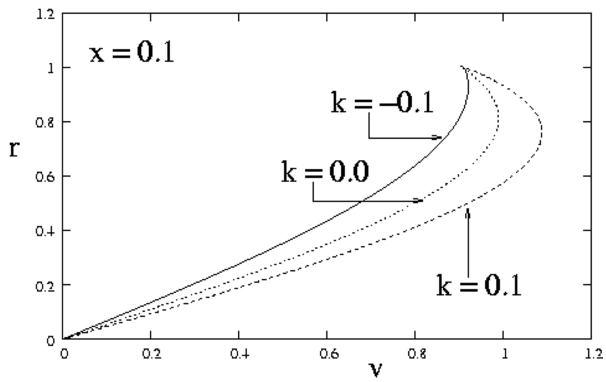

Figure-2(a). Distribution of radial velocity (v) with r . $\varepsilon = 0$, $\alpha = 1$, $x = 0.1$.

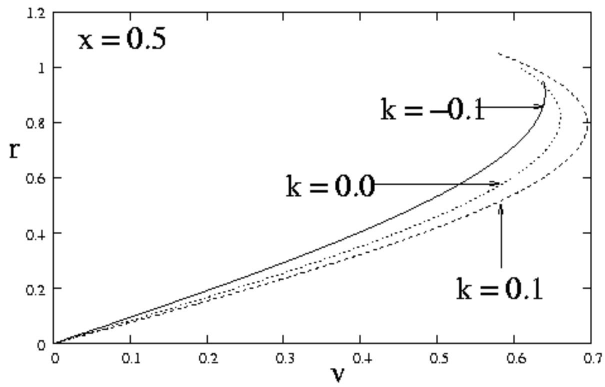

Figure-2(b). Distribution of radial velocity (v) with r . $\varepsilon = 0$, $\alpha = 1$, $x = 0.5$.

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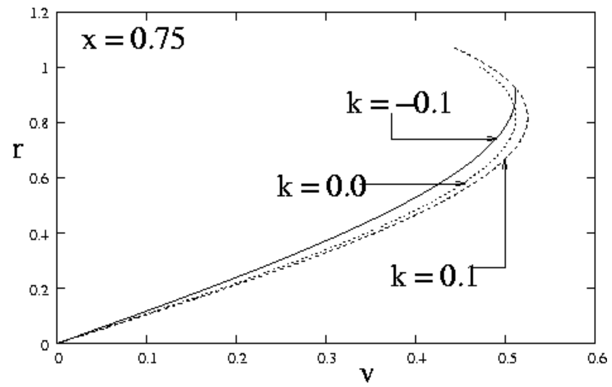

Figure-2(c). Distribution of radial velocity (v) with r.
$\varepsilon = 0,\quad \alpha = 1,\quad x = 0.75.$

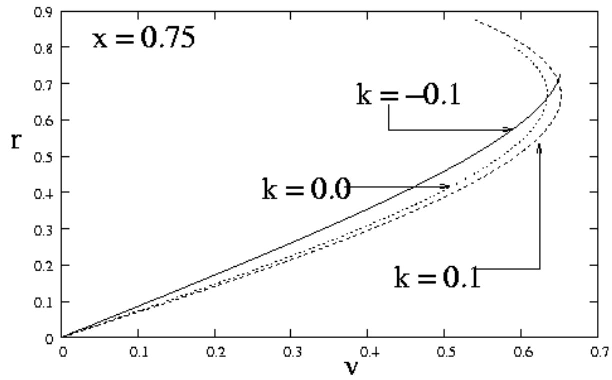

Figure-3(c). Distribution of radial velocity (v) with r.
$\varepsilon = 0.2,\quad \alpha = 1,\quad x = 0.75.$

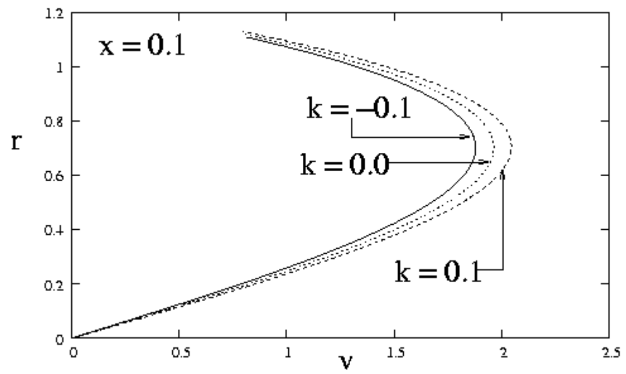

Figure-3(a). Distribution of radial velocity (v) with r.
$\varepsilon = 0.2,\quad \alpha = 1,\quad x = 0.1.$

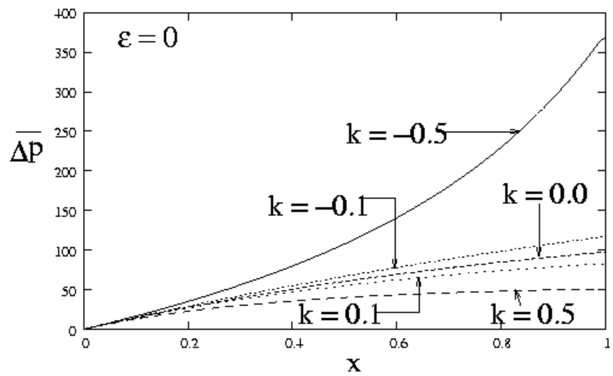

Figure-4(a). Distribution of mean pressure drop ($\overline{\Delta p}$) with x. $\varepsilon = 0.0,\quad \alpha = 1.0.$

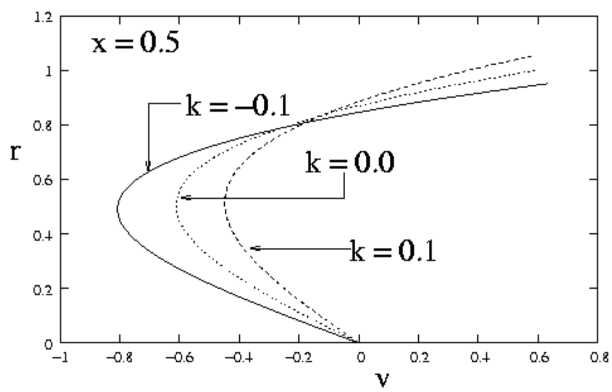

Figure-3(b). Distribution of radial velocity (v) with r.
$\varepsilon = 0.2,\quad \alpha = 1,\quad x = 0.5.$

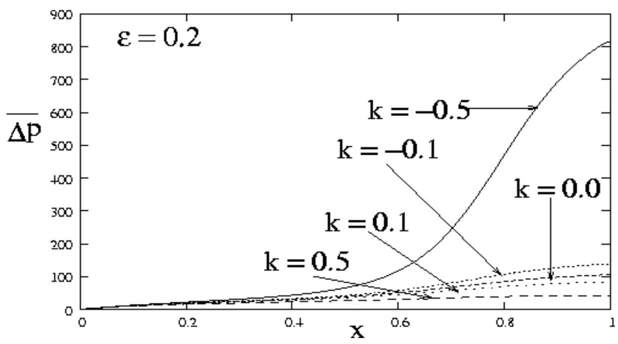

Figure-4(b). Distribution of mean pressure drop ($\overline{\Delta p}$) with x. $\varepsilon = 0.0,\quad \alpha = 1.0.$

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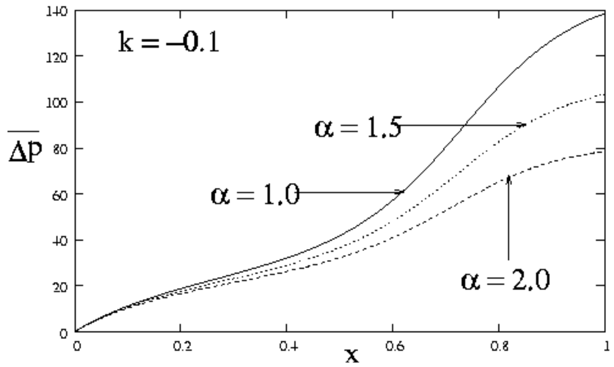

Figure-5(a). Distribution of mean pressure drop ($\overline{\Delta p}$) with x . $\varepsilon = 0.2,$ $k = -0.1$.

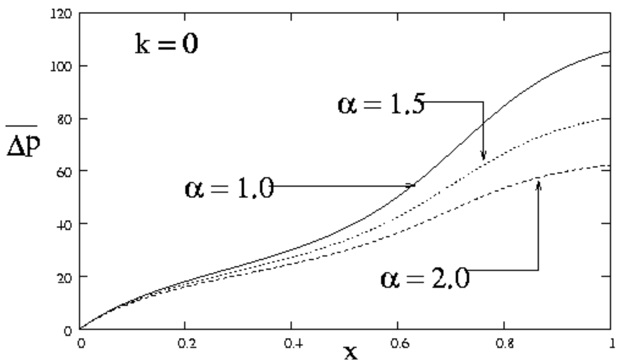

Figure-5(b). Distribution of mean pressure drop ($\overline{\Delta p}$) with x . $\varepsilon = 0.2,$ $k = 0.0$.

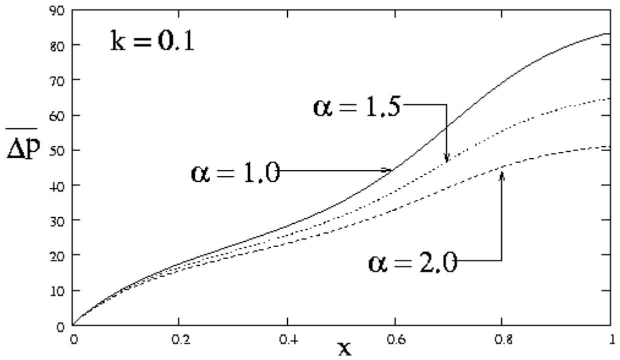

Figure-5(c). Distribution of mean pressure drop ($\overline{\Delta p}$) with x . $\varepsilon = 0.2,$ $k = 0.1$.

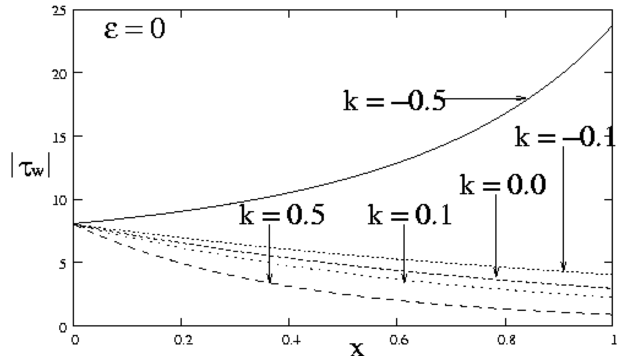

Figure-6(a). Distribution of wall shear stress ($|\tau_w|$) with x . $\varepsilon = 0.0,$ $\alpha = 1.0$.

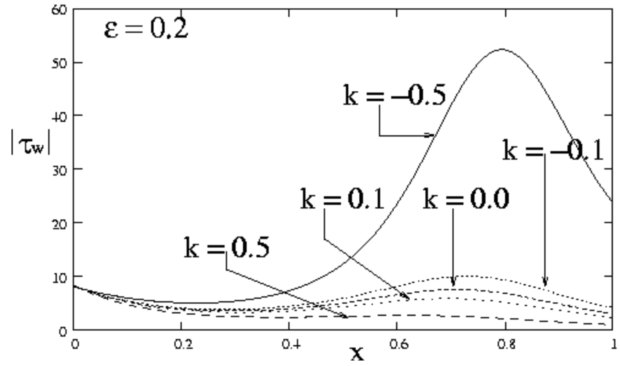

Figure-6(b). Distribution of wall shear stress ($|\tau_w|$) with x . $\varepsilon = 0.2,$ $\alpha = 1.0$.

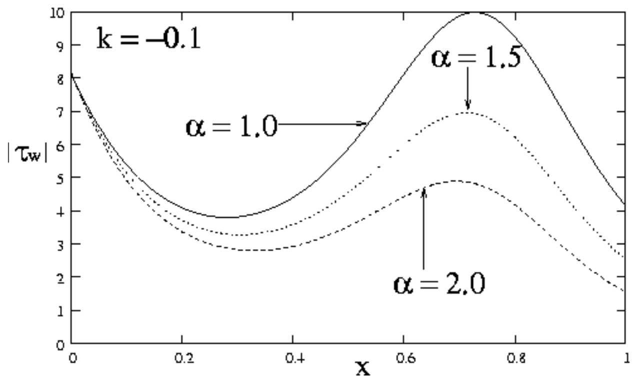

Figure-7(a). Distribution of wall shear stress ($|\tau_w|$) with x . $\varepsilon = 0.2,$ $k = -0.1$.

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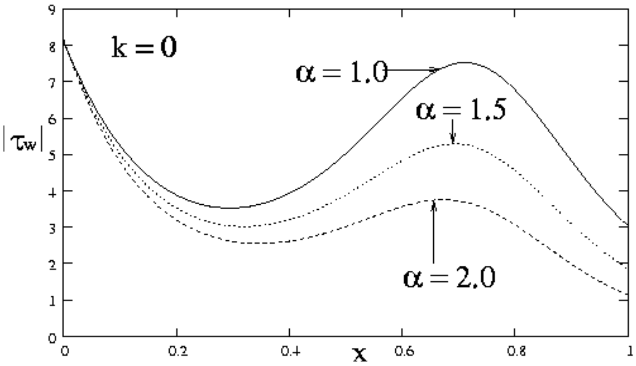

Figure-7(b). Distribution of wall shear stress ($|\tau_w|$) with x . $\varepsilon = 0.2,$ $k = 0.0$.

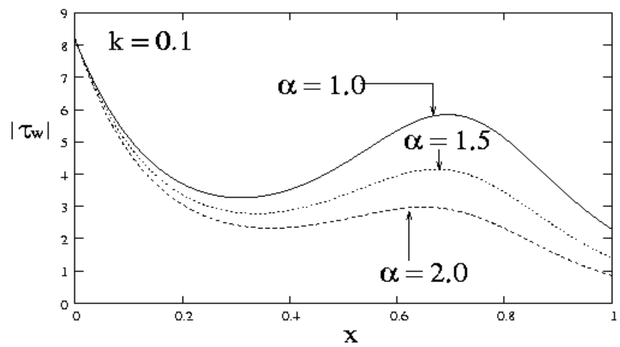

Figure-7(c). Distribution of wall shear stress ($|\tau_w|$) with x . $\varepsilon = 0.2,$ $k = 0.1$.

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