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A MODEL OF TWO MUTUALLY INTERACTING SPECIES WITH LIMITED RESOURCES OF FIRST SPECIES AND UNLIMITED FOR SECOND SPECIES

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ABSTRACT

The present paper concerns with a model of two mutually interacting species with limited resources for first species and unlimited resources for second species. The model is characterized by a coupled system of first order non-linear ordinary differential equations. In this case, we have identified two equilibrium points and described their stability criteria. Solutions for the linearized perturbed equations are also found and explained their significance. Under the limited and unlimited resources for first and second species respectively, if the death rate is greater than the birth rate for both the species, it is found that there are two equilibrium points. The stability criteria for these equilibrium points are derived and further the solutions of the linearized perturbed equations are found and illustrated.

Keywords: model, mutually interacting species, equilibrium points, mutualism, coexistence state, stability.

1. INTRODUCTION

Mathematical modeling of ecosystems was initiated by Lotka [1] and by Volterra [2]. The general concept of modeling has been presented in the treatises of Meyer [3], Cushing [4], Paul Colinvaux [5], Freedman [6], Kapur [7, 8]. The ecological interactions can be broadly classified as prey-predation, competition, mutualism and so on. N.C. Srinivas [9] studied the competitive eco-systems of two species and three species with regard to limited and unlimited resources. Later, Lakshmi Narayan [10] has investigated the two species prey-predator models. Recently stability analysis of competitive species was investigated by Archana Reddy [11]. Local stability analysis for a two-species ecological mutualism model has been presented by the present authors [12, 13]. Mutualism is any relationship between two species of organisms that benefits both species. Pollination (flowers and insects), seed dispersal (berries and fruit eaten by birds and animals), and lichens (fungus and algae) are examples for mutualism.

The present investigation is devoted to the analytical study of a model of two mutually interacting species with limited resources for first species and unlimited resources for second species. The model is characterized by a coupled pair of first order non-linear ordinary differential equations. Only two equilibrium points of the system are identified and the stability analysis is carried out. In case when death rate is greater than the birth rate for both the species, only two equilibrium points are identified and their stability criteria are derived. Solutions for the linearized perturbed equations are also found and explained their significance.

Before describing a model, first we make the following assumptions:

N_1 is the population of the first species, N_2, the population of the second species, a_1 is the rate of natural growth of the first species, a_2 is the rate of natural growth of the second species, α_{11} is the rate of decrease of the first species due to insufficient food, α_{12} is the rate of increase of the first species due to interaction with the second species, α_{21} is the rate of increase of the second species due to interaction with the first species. Further note that the variables N_1, N_2 and the model parameters $a_1, a_2, \alpha_{11}, \alpha_{12}, \alpha_{21}$ are non-negative and that the rate of difference between the death and birth rates is identified as the natural growth rate with appropriate sign. The model equations for a two species mutualising are governed by a system of non-linear ordinary differential equations.

2. BASIC EQUATIONS

The equation for the growth rate of first species (N_1) under limited resources is given by

$$\frac{dN_1}{dt} = a_1 N_1 - \alpha_{11} N_1^2 + \alpha_{12} N_1 N_2 \qquad (2.1)$$

The equation for the growth rate of second species (N_2) under unlimited resources is given by

$$\frac{dN_2}{dt} = a_2 N_2 + \alpha_{21} N_1 N_2 \qquad (2.2)$$

Before establishing the stability criteria, we shall make the distinction between various equilibrium states. The system under investigation has two equilibrium states.

I. $\overline{N}_1 = 0; \overline{N}_2 = 0$, the state in which both the species are washed out.

II. $\overline{N}_1 = \frac{a_1}{\alpha_{11}}; \overline{N}_2 = 0$, the state in which the first species (N_1) survives and the second species (N_2) is washed out.

Now we study the stability of these equilibrium states. Let us write

$N = (N_1, N_2) = \overline{N} + U$

Where U = (u_1, u_2) is a small perturbation over the equilibrium state $\overline{N} = (\overline{N}_1, \overline{N}_2)$.

The basic equations (2.1), (2.2) are linearized to obtain the equations for the perturbed state,

$$\frac{dU}{dt} = AU \qquad (2.3)$$

Where

$$A = \begin{bmatrix} a_1 - 2\alpha_{11}\overline{N}_1 + \alpha_{12}\overline{N}_2 & \alpha_{12}\overline{N}_1 \\ \alpha_{21}\overline{N}_2 & a_2 + \alpha_{21}\overline{N}_1 \end{bmatrix} \qquad (2.4)$$

The characteristic equation for the system is

$$det\left[A - \lambda I\right] = 0 \qquad (2.5)$$

The equilibrium state is stable, if both the roots of the equation (2.5) are negative in case they are real or have negative real parts in case they are complex.

Equilibrium state I (fully washed out state):

To discuss the stability of equilibrium state $\overline{N}_1 = 0; \overline{N}_2 = 0$, we consider small perturbations $u_1(t)$ and $u_2(t)$ from the steady state, i.e. we write

$$N_1 = \overline{N}_1 + u_1(t), \qquad (2.6)$$

$$N_2 = \overline{N}_2 + u_2(t). \qquad (2.7)$$

Substituting (2.6) and (2.7) in (2.1) and (2.2), we get

$$\frac{du_1}{dt} = a_1u_1 - \alpha_{11}u_1^2 + \alpha_{12}u_1u_2$$

$$\frac{du_2}{dt} = a_2u_2 + \alpha_{21}u_1u_2$$

After linearization, we get

$$\frac{du_1}{dt} = a_1u_1 \qquad (2.8)$$

and

$$\frac{du_2}{dt} = a_2u_2 \qquad (2.9)$$

The characteristic equation is

$(\lambda - a_1)(\lambda - a_2) = 0$,

whose roots a_1, a_2 are both positive. Hence the equilibrium state is unstable.

The solutions of equations (2.8) and (2.9) are

$$u_1 = u_{10}\, e^{a_1 t} \qquad (2.10)$$

$$u_2 = u_{20}\, e^{a_2 t} \qquad (2.11)$$

Where u_{10}, u_{20} are the initial values of u_1 and u_2. The solution curves are illustrated in Figures 1 to 4

Case 1: $a_1 < a_2$ and $u_{10} < u_{20}$ i.e. the second species dominates the first species in the natural growth rate as well as in its initial population strength.

In this case, the second species continues out numbering the first species as shown in Figure-1.

Case 2: $a_1 < a_2$ and $u_{10} > u_{20}$ i.e. the second species dominates the first species in the natural growth rate but its initial strength is less than that of first species.

In this case, the first species out numbers the second species till the time,

$$t = t^* = \frac{ln\ \{u_{10} / u_{20}\}}{(a_2 - a_1)}$$

after that the second species out numbers the first species.

Case 3: $a_1 > a_2$ and $u_{10} < u_{20}$ i.e. the first species dominates the second species in the natural growth rate but its initial strength is less than that of second species.

In this case, the second species out numbers the first species till the time,

$$t = t^* = \frac{ln\ \{u_{10} / u_{20}\}}{(a_2 - a_1)}$$

after that the first species out numbers the second species.

Case 4: $a_1 > a_2$ and $u_{10} > u_{20}$ i.e. the first species dominates the second species in the natural growth as well as in its initial population strength.

In this case, the first species continues out numbering the second species as shown in Figure-4. Further the trajectories in the (u_1, u_2) plane are given by

$$\left[\frac{u_1}{u_{10}}\right]^{a_2} = \left[\frac{u_2}{u_{20}}\right]^{a_1}$$

and these are illustrated in Figure-5.

Equilibrium state II (N_1 exists while N_2 is washed out):

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We have

$$\overline{N}_1 = \frac{a_1}{\alpha_{11}}; \overline{N}_2 = 0$$

Substituting (2.6) and (2.7) in (2.1) and (2.2), we get

$$\frac{du_1}{dt} = -a_1u_1 - \alpha_{11}u_1^2 + \alpha_{12}u_1u_2 + \frac{a_1\alpha_{12}u_2}{\alpha_{11}}$$

$$\frac{du_2}{dt} = a_2u_2 + \alpha_{21}u_1u_2 + \frac{a_1\alpha_{21}u_2}{\alpha_{11}}$$

After linearization, we get

$$\frac{du_1}{dt} = -a_1u_1 + \frac{a_1\alpha_{12}u_2}{\alpha_{11}} \quad (2.12)$$

and

$$\frac{du_2}{dt} = \left[a_2 + \frac{a_1\alpha_{21}}{\alpha_{11}}\right]u_2 \quad (2.13)$$

The characteristic equation is

$$(\lambda + a_1)\ \{\lambda - [a_2 + \frac{a_1\alpha_{21}}{\alpha_{11}}]\}$$

One root of this equation is $\lambda_1 = -a_1$ which is negative while the other root is

$\lambda_2 = a_2 + \frac{a_1\alpha_{21}}{\alpha_{11}}$ which is positive. Hence the equilibrium state is unstable.

The trajectories are given by

$$u_1 = \frac{1}{\gamma_1}\left[u_{20}a_1\alpha_{12}e^{\lambda_2 t} + \{u_{10}\gamma_1 - u_{20}a_1\alpha_{12}\}e^{-a_1 t}\right] \quad (2.14)$$

$$u_2 = u_{20}e^{\lambda_2 t} \quad (2.15)$$

Where

$$\lambda_2 = a_2 + \frac{a_1\alpha_{21}}{\alpha_{11}};\ \gamma_1 = a_2\alpha_{11} + a_1[\alpha_{11} + \alpha_{21}]$$

The solution curves are illustrated in Figures 6 and 7

CASE 1: $u_{10} < u_{20}$ i.e. the second species dominates the first species in its initial strength.

We notice that the second species is going away from the equilibrium point while the first species would become extinct at the instant

$$t_1^* = \frac{1}{(\lambda_2 + a_1)} ln\left[\frac{u_{20}\alpha_{12}a_1 - u_{10}\gamma_1}{u_{20}\alpha_{12}a_1}\right]$$

As such the state is unstable.

CASE 2: $u_{10} > u_{20}$ i.e. the initial strength of first species is greater than that of the second species.

Initially the first species out numbers the second species and this continues up to the time instant,

$$t = t^* = \frac{1}{\lambda_2 + a_1} ln\left\{\frac{u_{10}\alpha_{11}(\lambda_2 + a_1) - u_{20}a_1\alpha_{12}}{u_{20}[\alpha_{11}(\lambda_2 + a_1) - a_1\alpha_{12}]}\right\}$$

there after the second species out numbers the first species. And also the second species is noted to be going away from the equilibrium point while the first species would become extinct at the instant

$$t_1^* = \frac{1}{(\lambda_2 + a_1)} ln\left[\frac{u_{20}\alpha_{12}a_1 - u_{10}\gamma_1}{u_{20}\alpha_{12}a_1}\right]$$

As such the state is unstable. Also the trajectories in the (u_1, u_2) plane are given by

$$(q_1 - 1)u_1 = c\,u_2^{\,q_1} - p_1u_2$$

Where

$$p_1 = \frac{a_1\alpha_{12}}{a_1\alpha_{21} + a_2\alpha_{11}};$$

$$q_1 = \frac{-a_1\alpha_{11}}{a_1\alpha_{21} + a_2\alpha_{11}}$$

and c is an arbitrary constant.

The solution curves are illustrated in Figure-8.

3. THE DEATH RATE IS GREATER THAN THE BIRTH RATE FOR BOTH THE SPECIES

The basic equations governing the system are

$$\frac{dN_1}{dt} = -a_1N_1 - \alpha_{11}N_1^2 + \alpha_{12}N_1N_2 \quad (3.1)$$

$$\frac{dN_2}{dt} = -a_2N_2 + \alpha_{21}N_1N_2 \quad (3.2)$$

Here we come across two equilibrium states:

I. $\overline{N}_1 = 0; \overline{N}_2 = 0$ (3.3)

The state in which both the species are washed out .

II. $\overline{N}_1 = \frac{a_2}{\alpha_{21}}; \overline{N}_2 = \frac{a_1\alpha_{21} + a_2\alpha_{11}}{\alpha_{12}\alpha_{21}}$ (3.4)

The state in which both the species co-exist.

Equilibrium state I (fully washed out state):

To discuss the stability of equilibrium state $\overline{N}_1 = 0 ; \overline{N}_2 = 0$, we consider small perturbations $u_1(t)$ and $u_2(t)$ from the steady state, i.e. we write

$$N_1 = \overline{N}_1 + u_1(t), \tag{3.5}$$

$$N_2 = \overline{N}_2 + u_2(t). \tag{3.6}$$

Substituting (3.5) and (3.6) in (3.1) and (3.2), we get

$$\frac{du_1}{dt} = -a_1 u_1 - \alpha_{11} u_1^2 + \alpha_{12} u_1 u_2$$

$$\frac{du_2}{dt} = -a_2 u_2 + \alpha_{21} u_1 u_2$$

After linearization, we get

$$\frac{du_1}{dt} = -a_1 u_1 \tag{3.7}$$

and

$$\frac{du_2}{dt} = -a_2 u_2 \tag{3.8}$$

The characteristic equation is

$(\lambda + a_1)(\lambda + a_2) = 0$

The roots of this equation, $\lambda_1 = -a_1$ and $\lambda_2 = -a_2$ are both negative. Hence the equilibrium state is stable. The solutions of equations (3.7) and (3.8) are

$$u_1 = u_{10}\, e^{-a_1 t} \tag{3.9}$$

$$u_2 = u_{20}\, e^{-a_2 t} \tag{3.10}$$

Where u_{10}, u_{20} are the initial values of u_1 and u_2. The solution curves are illustrated in Figures 9 to 12.

CASE 1: $a_1 > a_2$ and $u_{10} > u_{20}$ i.e., the first species dominates the second species in the natural growth rate as well as in its initial population strength.

In this case the first species continues out numbering the second species as shown in Figure-9. It is evident that both the species converging asymptotically to the equilibrium point. Hence the equilibrium state is stable.

CASE 2: $a_1 > a_2$ and $u_{10} < u_{20}$ i.e., the first species dominates the second species in the natural growth rate but its initial strength is less than that of second species.

In this case, initially the second species out numbers the first species and this continues up to the time,

$$t = t^* = \frac{ln\ \{u_{10} / u_{20}\}}{(a_1 - a_2)}$$

after that the first species out numbers the second species. As t $\rightarrow \infty$ both u_1 and u_2 approach to the equilibrium point. Hence the state is stable.

CASE 3: $a_1 < a_2$ and $u_{10} < u_{20}$ i.e. the second species dominates the first species in the natural growth rate as well as in its initial population strength.

In this case the second species always out numbers the first species. It is evident that both the species converging asymptotically to the equilibrium point. Hence the state is stable.

CASE 4: $a_1 < a_2$ and $u_{10} > u_{20}$ i.e., the second species dominates the first species in the natural growth rate but its initial strength is less than that of first species.

In this case, the first species dominates the second species till the time,

$$t = t^* = \frac{ln\ \{u_{10} / u_{20}\}}{(a_1 - a_2)}$$

after that the second species out numbers the first species. As t $\rightarrow \infty$ both u_1 and u_2 approach to the equilibrium point. Hence the state is stable. Also the trajectories in the (u_1, u_2) plane are given by

$$\left[\frac{u_1}{u_{10}}\right]^{-a_2} = \left[\frac{u_2}{u_{20}}\right]^{-a_1}$$

Equilibrium state II (coexistence state):

We have

$$\overline{N}_1 = \frac{a_2}{\alpha_{21}} ; \overline{N}_2 = \frac{a_2\alpha_{11} + a_1\alpha_{21}}{\alpha_{12}\alpha_{21}}$$

Substituting (3.5) and (3.6) in (3.1) and (3.2), we get

$$\frac{du_1}{dt} = -\alpha_{11} u_1^2 + \alpha_{12} u_1 u_2 - \alpha_{11} \overline{N}_1 u_1 + \alpha_{12} \overline{N}_1 u_2$$

$$\frac{du_2}{dt} = \alpha_{21} u_1 \overline{N}_2 + \alpha_{21} u_1 u_2$$

After linearization, we get

$$\frac{du_1}{dt} = -\alpha_{11} \overline{N}_1 u_1 + \alpha_{12} \overline{N}_1 u_2 \tag{3.11}$$

and

$$\frac{du_2}{dt} = \alpha_{21} u_1 \overline{N}_2 \tag{3.12}$$

The characteristic equation is

$$\lambda^2 + \alpha_{11}\overline{N}_1 \lambda - \alpha_{12}\alpha_{21}\ \overline{N}_1 \overline{N}_2 = 0$$

One root of this equation is positive and the other root is negative. Hence the equilibrium state is unstable. The trajectories are given by

$$u_1 = \left[\frac{u_{10}\lambda_1 + u_{20}\alpha_{12}\overline{N}_1}{\lambda_1 - \lambda_2}\right] e^{\lambda_1 t} + \left[\frac{u_{10}\lambda_2 + u_{20}\alpha_{12}\overline{N}_1}{\lambda_2 - \lambda_1}\right] e^{\lambda_2 t}$$

$$u_2 = \left[\frac{u_{20}(\lambda_1 + \alpha_{11}\overline{N}_1) + u_{10}\alpha_{21}\overline{N}_2}{\lambda_1 - \lambda_2}\right] e^{\lambda_1 t} + \left[\frac{u_{20}(\lambda_2 + \alpha_{11}\overline{N}_1) + u_{10}\alpha_{21}\overline{N}_2}{\lambda_2 - \lambda_1}\right] e^{\lambda_2 t}$$

The curves are illustrated in Figures 13 and 14.

Case 1: $u_{10} > u_{20}$ i.e. initially the first species dominates the second species.

In this case, the first species is noted to be going away from the equilibrium point while the second species approaches asymptotically to the equilibrium point. Hence the state is unstable.

Case 2: $u_{10} < u_{20}$ i.e., initially the second species dominates the first species.

In this case the second species dominates the first species till the time,

$$t = t^* = \frac{1}{\lambda_2 - \lambda_1} \ln\left[\frac{(b_2-\lambda_1)u_{10} + (a_3 - b_1)u_{20}}{(b_2-\lambda_2)u_{10} + (a_4 - b_1)u_{20}}\right]$$

Where

$$b_1 = \alpha_{12}\overline{N}_1; \quad b_2 = \alpha_{21}\overline{N}_2;$$

$$a_3 = \lambda_1 + \alpha_{11}\overline{N}_1; \quad a_4 = \lambda_2 + \alpha_{11}\overline{N}_1$$

after that the first species dominates the second species and grows indefinitely while the second species asymptotically approaches to the equilibrium point. Hence the state is unstable. Further the trajectories in the (u_1, u_2) plane are given by

$$[u_2^{(a-1)(v_1-v_2)}]d = \frac{(u_1 - u_2 v_1)^{a v_1}}{(u_1 - v_2 u_2)^{a v_2}}$$

where v_1 and v_2 are roots of the quadratic equation $av^2 + bv + c = 0$ with $a = \alpha_{21}\overline{N}_2$; $b = \alpha_{11}\overline{N}_1$; $c = -\alpha_{12}\overline{N}_1$ and d is an arbitrary constant.

4. TRAJECTORIES

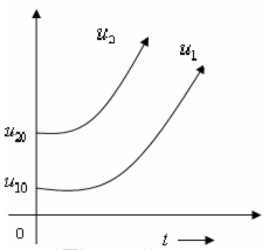


Figure 1

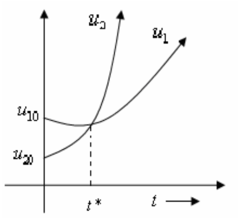


Figure 2

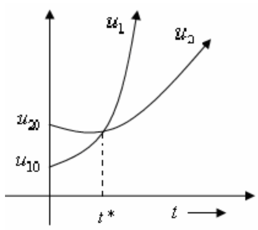


Figure 3

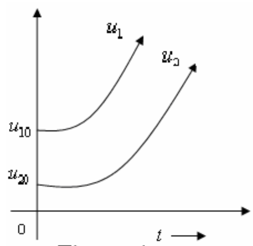


Figure 4

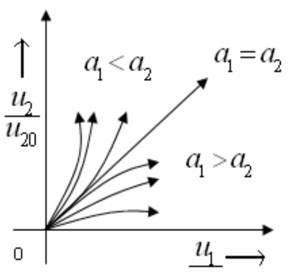
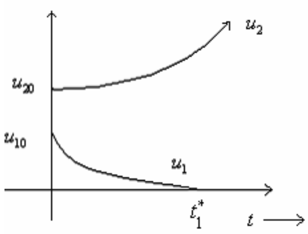


Figure.5



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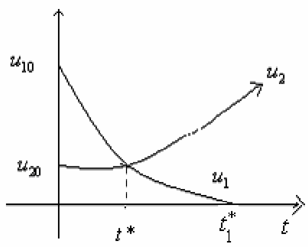

Figure 7

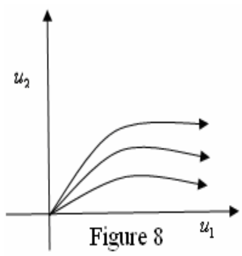

Figure 8

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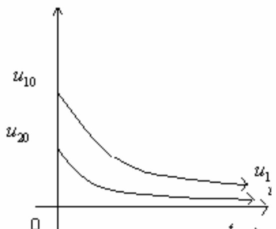

Figure 9

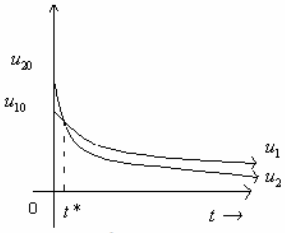

Figure 10

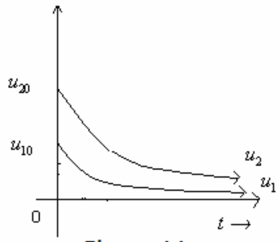

Figure 11

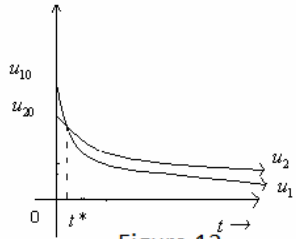

Figure 12

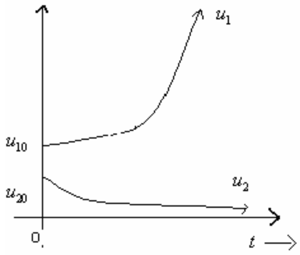

Figure 13

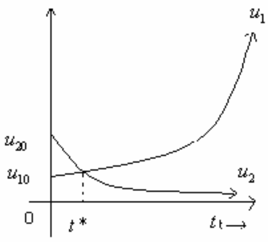

Figure 14

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