VOL. 6, NO. 2, FEBRUARY 2011 ISSN 1819-6608
ARPN Journal of Engineering and Applied Sciences

www.arpnjournals.com

A TWO SPECIES COMMENSALISM MODEL WITH LIMITED RESOURCES- A NUMERICAL APPROACH

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ABSTRACT

In this paper a mathematical model of commensalism between two species (S_1 and S_2) with limited resources is investigated. The commensal species (S_1), in spite of the limitation of its natural resources flourishes drawing strength from the host species (S_2). This model is characterized by a pair of first order non-linear coupled differential equations. All the four equilibrium points of the model are identified and the criteria for the stability are discussed. The non-linear coupled system-equations are solved numerically by employing fourth order Runge-Kutta method and the results are presented. Further, some threshold results are stated.

Keywords: mathematical model, commensalism, species, differential equations, Runge-Kutta method, stability analysis.

1. INTRODUCTION

Ecology relates to the study of living beings in relation to their living styles. Research in the area of theoretical ecology was initiated by Lotka [8] and by Volterra [17]. Since then many mathematicians and ecologists contributed to the growth of this area of knowledge as reported in the treatises of Meyer [9], Kushing [5], Kapur [3, 4] etc. The ecological interactions can be broadly classified as Prey - predation, competition, commensalism, Ammensalism, Neutralism and so on. N.C. srinivas [16] studied competitive eco-systems of two species and three species with limited and unlimited resources. Later, Lakshminarayan [6], Lakshminarayan and Pattabhiramacharyulu [7] studied Prey-predator ecological models with a partial cover for the prey and alternate food for the predator. Recently, stability analysis of competitive species was carried out by Archana Reddy, Paattabh Rramacharyulu and Gandhi [1] and by Bhaskara sarma and Pattabhi Ramacharyulu [2], while Ravindra Reddy [14] investigated mutualism between two species phanikumar [10], Phanikumar, Seshagiri Rao, Pattabhi Ramacharyulu [11,12], phanikumar, Pattabhi Ramacharyulu [13,14], Phanikumar, Ravindrareddy, Pattabhi Ramacharyulu [15] investigated commensalism between two species.

The present investigation is devoted to the analytical study of commensalism between two species. A two species Commensalisms is an ecological relationship between two species where one species (S_1) derives a benefit from the other (S_2) which does not get affected by it: S_1 may be referred as the commensal species while S_2 the host. Some examples are Cattle Egret, Anemonetish, and Barnacles etc.

The host species (S_2) supports the commensal species (S_1) which has a natural growth rate in spite of a support other than from S_1. The commensal species (S_1), in spite of the limitation of its natural resources flourishes drawing strength from the host species (S_2). The model is characterized by a coupled pair of first order non-linear differential equations. In all four equilibrium points of the system are identified and the stability analysis is carried out. It is noticed that the co-existent state is the only one stable state. All other three equilibrium states are unstable.

2. BASIC EQUATIONS

Notations adopted

N_1, N_2: The populations of the commensal (S_1) and host (S_2) species, respectively at time t

$K_i = \dfrac{a_i}{a_{ii}}$: Carrying capacities of S_i, i = 1, 2 (these parameters characterize the amount of resources available for the consumption exclusively for the two species.)

$C = \dfrac{a_{12}}{a_{11}}$: Commensalism coefficient.

Further both the variables N_1 and N_2 are non-negative and the model parameters a_1, a_2, a_{11}, a_{22}, and a_{12} are assumed to be non-negative constants.

Employing the above terminology, the model equations for a two species commensaling system are given by the following system of non- linear differential equations.

Equations for the growth rate of the commensal species (S_1) and host species (S_2) can be written as follows:

$$\frac{dN_1}{dt} = a_{11}N_1\left[K_1 - N_1 + CN_2\right] \tag{1}$$

$$\frac{dN_1}{dt} = a_{22}N_2\left[K_2 - N_2\right]$$

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3. MATHEMATICAL ANALYSIS

3.1 Existence of equilibrium points

In order to find out the equilibrium points of the system (1), we set $\dot{N}_1 = \dot{N}_2 = 0$ where $\dot{N}_i = \frac{dN_i}{dt}$, $i = 1, 2$ (2)

We find that there are four non-negative equilibrium points namely E_1 (0, 0), E_2 (0, K_2), E_3 (K_1, 0), E_4 (K_1+ CK_2, K_2). The point E_1 is the trivial equilibrium point i.e. both the species are washed out. The points E_2 and E_3 in which commensal and host species are washed out, respectively. The point E_4 is the nontrivial equilibrium point i.e. both the species are exists.

3.2 Stability analysis

The local stability of each equilibrium point can be can be studied by computing the corresponding variational matrix,

$$A = \begin{bmatrix} a_1 - 2a_{11}\overline{N_1} + a_{12}\overline{N_2} & a_{12}\overline{N_1} \\ 0 & a_2 - 2a_{22}\overline{N_2} \end{bmatrix}$$

From variational matrix analysis, following points regarding the local stability of equilibria can be concluded:

a) E_1 is unstable.
b) E_2 is unstable.
c) E_3 is unstable.
d) E_4 is stable.

4. A NUMERICAL SOLUTION OF THE BASIC NON-LINEAR COUPLED DIFFERENTIAL EQUATIONS: THE VARIATION OF N_1 AND N_2 VERSES TIME't'.

The variation of N_1 and N_2 verses time t in the interval [0, 5] is computed numerically employing 4th order Runge-Kutta system for a wide range of values of the characterizing parameters a_1, a_2; a_{11}, a_{22}; a_{12} as shown in Table-1. For this MATLAB has been used and the results are illustrated in Figures 4.1 to 4.10.

Table-1

S. No.	a_1	a_{11}	A_{12}	a_2	a_{22}	N_{10}	N_{20}	Figure No. showing variation of N_1 N_2 vs. 't'
1	1.5	0.05	0.02	1.7	0.007	2	1	4.1
2	1.35	0.02	0.002	1.9	0.02	2	1.5	4.2
3	2.3	0.04	0.1	2.5	0.05	3	2	4.3
4	2.5	0.6	0.005	1.5	0.3	1.5	1	4.4
5	1	0.007	0.003	2.25	0.03	2	3	4.5
6	2	0.1	0.007	1.3	0.08	1	2	4.6
7	1.2	0.009	0.0002	0.8	0.006	3	1.7	4.7
8	0.25	0.004	0.05	0.75	0.003	2	0.5	4.8
9	1	0.9	0.7	1.5	0.5	2	1	4.9
10	2.5	0.32	0.0006	2	0.15	2	0.7	4.10

Case1:

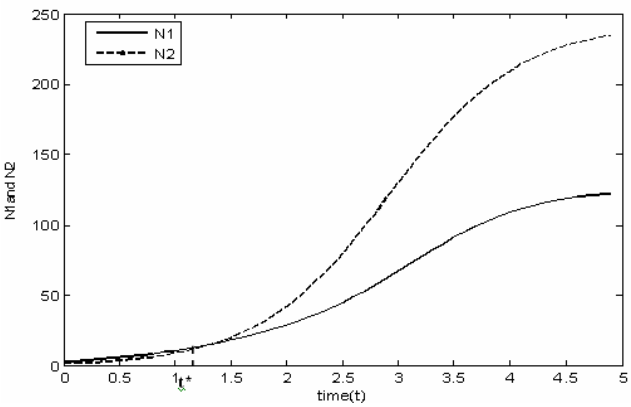

Figure-4.1. Variation of N_1, N_2 vs. t for a_1=1.5, a_{11}= 0.05, a_{12}=0.02, a_2=1.7, a_{22}=0.007, N_{10}=2, N_{20}=1.

In case 1 the first species outnumbers the second species up to a time t =t*=1.4 after which the second species dominates the first species. Further, there is no appreciable growth in the first species whereas the second species has a steep rise, as seen in Figure-4.1.

In case 2 the first species dominates the second species up to a time t= t*=1.2 after which the second species surpasses the first species. Further we notice that both the species increase and coexist after t*=1.2 with steady increase. This is illustrated in Figure-4.2.

In case 3, initially the first species outnumbers the second species and continuous to do so. Also we notice that there is a steep rise in the first species while the second species becomes extinct at t*=3.2. Further we observe that after the extinction of the second species the first species maintains a steady growth. This is illustrated in Figure-4.3

Case 2:

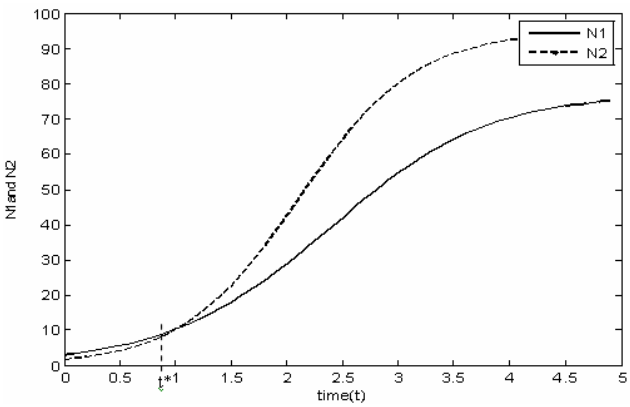

Figure-4.2. Variation of N_1, N_2 vs. t for a_1=1.35, a_{11}= 0.02, a_{12}=0.002, a_2=1.9, a_{22}=0.02, N_{10}=2, N_{20}=1.5.

Case 3:

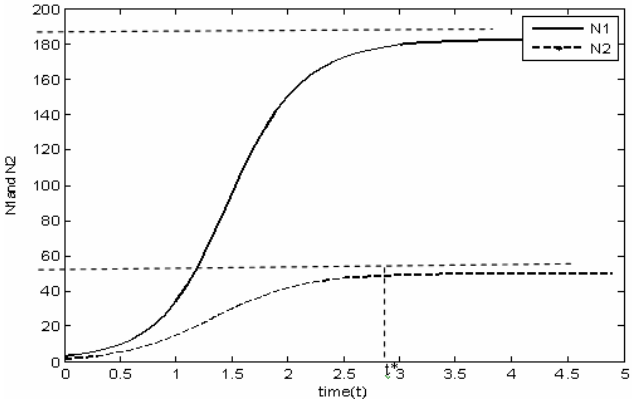

Figure-4.3. Variation of N_1, N_2 vs. t for a_1=2.3, a_{11}= 0.04, a_{12}=0.1, a_2=2.5, a_{22}=0.05, N_{10}=3, N_{20}=2.

Case 4:

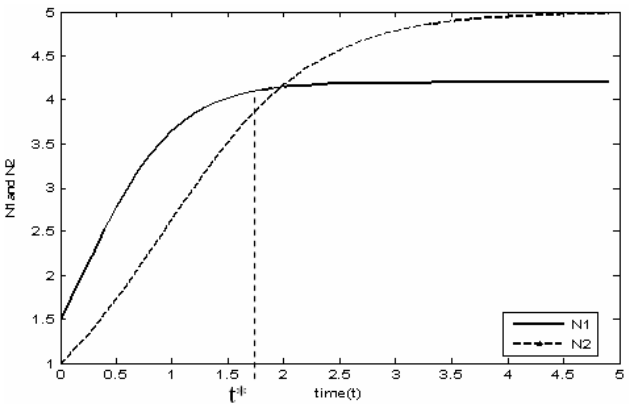

Figure-4.4. Variation of N_1, N_2 vs. t for a_1=2.5, a_{11}= 0.6, a_{12}=0.005, a_2=1.5, a_{22}=0.3, N_{10}=1.5, N_{20}=1.

In this case the first species dominates the second species up to a time t=t*=2, after which the second species surpasses the first species. Further we notice that both the species survive as illustrated in Figure-4.4.

Case 5:

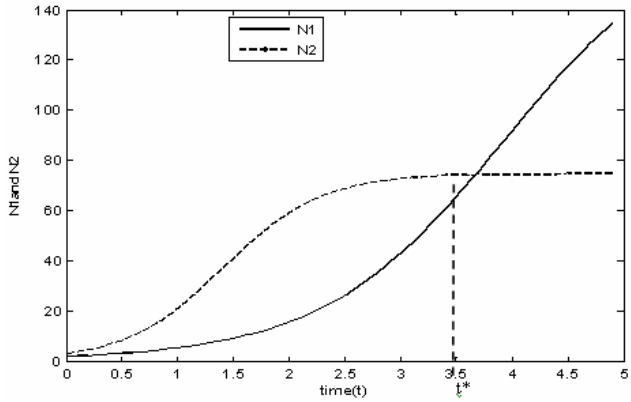

Figure-4.5. Variation of N_1, N_2 vs. t for a_1=1, a_{11}= 0.007, a_{12}=0.003, a_2=2.25, a_{22}=0.03, N_{10}=2, N_{20}=3.

Initially the second species out numbers the first species up to a time t=t*=3.6, after which the second species surpasses the first species. Further we notice that there is a steep rise in the species and the second species gradually becomes extinct.

Case6:

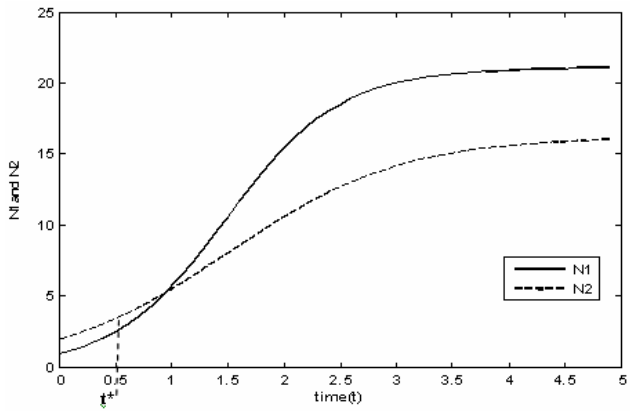

Figure-4.6. Variation of N_1, N_2 vs. t for a_1=2, a_{11}= 0.1, a_{12}=0.007, a_2=1.3, a_{22}=0.08, N_{10}=1 N_{20}=2.

Initially the second species dominates the first species up to a time t =t*=0.9. The dominance reversal time t* is shown in Figure-4.6 after which the first species dominates. Further we notice that both the species gradually become extinct.

In case 7, initially the first species is dominant over the second species and continues to do so. Also we notice that both the species are increasing as seen in Figure-4.7.

In case 8, initially the first species dominates the second species up to time t= t*=1.8 after which the second species dominates up to time t= t*= 4.3 after which the first species dominates. In this case we notice that initially

both the species are very low grow rate, after t*=4.3 both the species are increasing as seen in Figure-4.8.

Case 7:

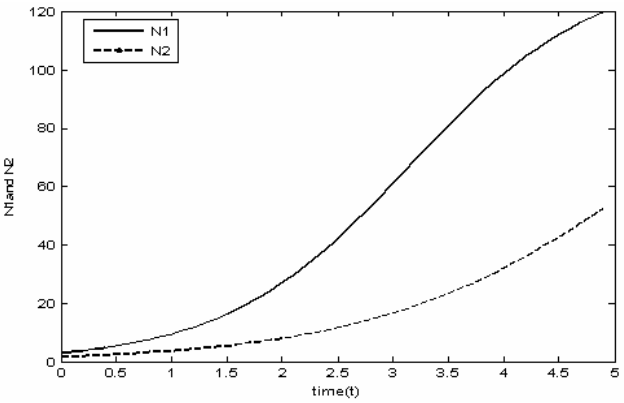

Figure-4.7. Variation of N_1, N_2 vs. t for a_1=1.2, a_{11}= 0.009, a_{12}=0.0002, a_2=0.8, a_{22}=0.006, N_{10}=3 N_{20}=1.7.

Case 8:

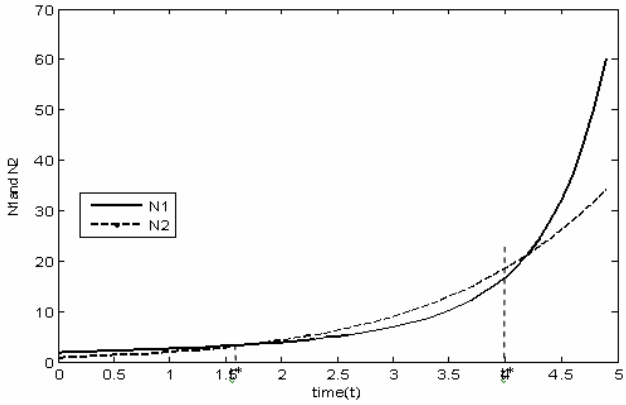

Figure-4.8. Variation of N_1, N_2 vs. t for a_1=0.25, a_{11}= 0.004, a_{12}=0.05, a_2=0.75, a_{22}=0.003, N_{10}=2, N20=0.5.

Case 9:

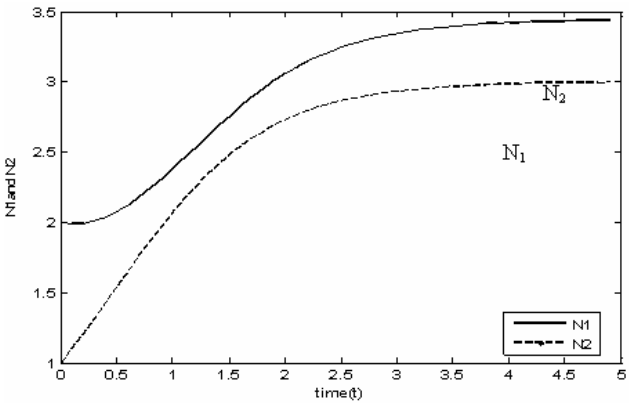

Figure-4.9. Variation of N_1, N_2 vs. t for a_1=1, a_{11}= 0.9, a_{12}=0.7, a_2=1.5, a_{22}=0.5, N_{10}=2 N_{20}=1.

In this case the first species would always dominate over the second species. Further we notice that both the species have a steady variation with no appreciable growth rates. This is seen in Figure-4.9.

Case 10:

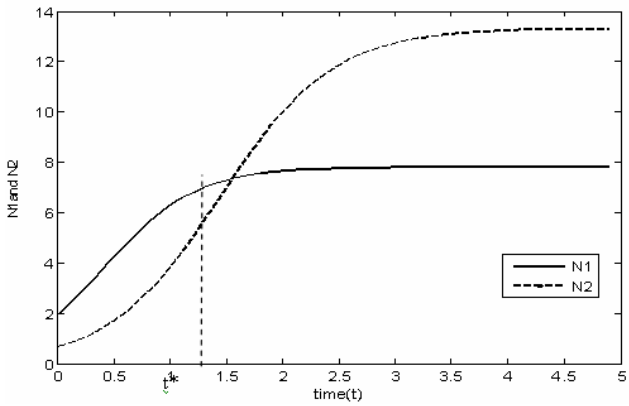

Figure-4.10. Variation of N_1, N_2 vs. t for a_1=2.5, a_{11}= 0.32, a_{12}=0.0006, a_2=2, a_{22}=0.15, N_{10}=2, N_{20}=0.7.

In this case initially the first species dominate, over the second species up to a time t =t*=1.5 after which the second species surpasses the first species. The rise in the second species is steep whereas there is a slight variation in the growth of the first species. This is illustrated in Figure-4.10.

5. THRESHOLD (OR) PHASE- PLANE DIAGRAM

The conditions $\frac{dN_1}{dt}=0$ and $\frac{dN_2}{dt}=0$ imply that neither N_1 nor N_2 changes its density .When we impose these conditions the basic equations give rise to two straight lines. At the points where, $\frac{dN_1}{dt}=0$, $\frac{dN_2}{dt}=0$ the resulting straight lines divide the phase plane into four regions (Figure-4.11).

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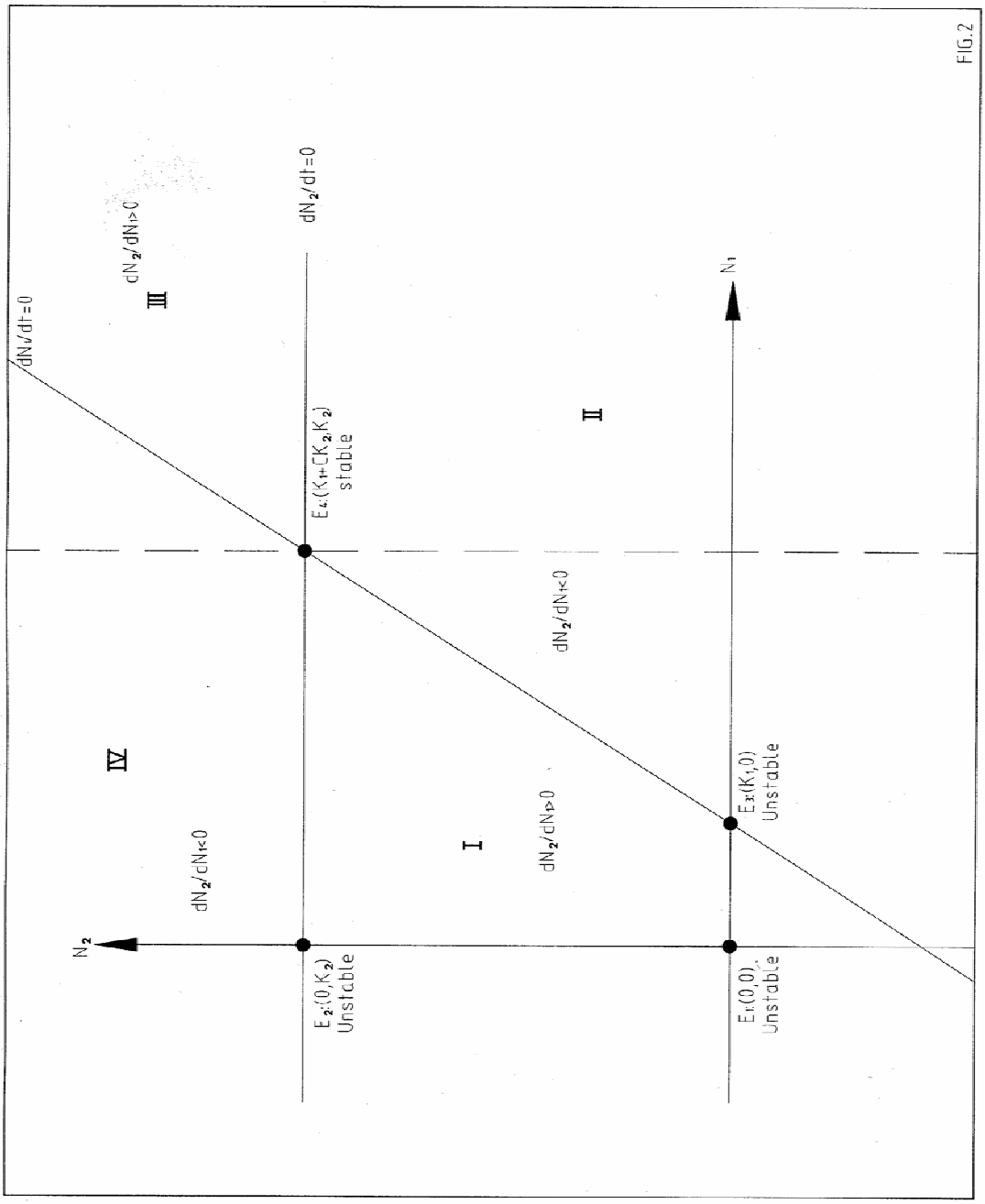

Figure-4.11. Threshold regions.

Region I: Both the species N_1 and N_2 flourish with time t.
Region II: The commensal species N_1 declines and the host species N_2 flourishes with time t.
Region III: Both the species N_1 and N_2 decline with time t.
Region IV: The commensal species N_1 flourish and the host species N_2 declines with time t.

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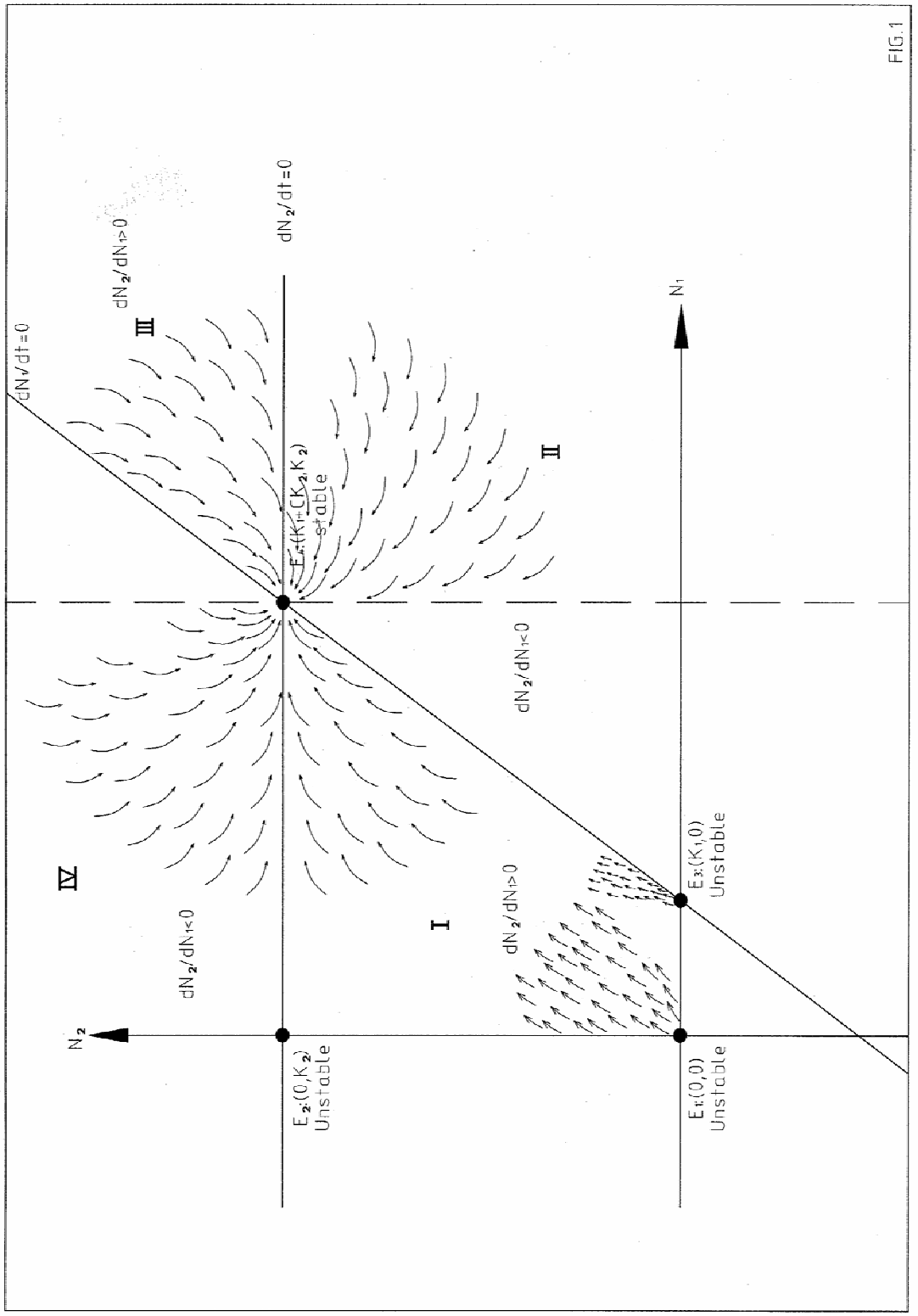

Figure-4.12. Threshold diagram.

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