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FINITE DIFFERENCE ANALYSIS OF CURVED DEEP BEAMS ON WINKLER FOUNDATION

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ABSTRACT

This research deals with the linear elastic behavior of curved deep beams resting on elastic foundations with both compressional and frictional resistances. Timoshenko's deep beam theory is extended to include the effect of curvature and the externally distributed moments under static conditions. As an application to the distributed moment generations, the problems of deep beams resting on elastic foundations with both compressional and frictional restraints have been investigated in detail. The finite difference method was used to represent curved deep beams and the results were compared with other methods to check the accuracy of the developed analysis. Several important parameters are incorporated in the analysis, namely, the vertical subgrade reaction, horizontal subgrade reaction, beam width, and also the effect of beam thickness to radius ratio on the deflections, bending moments and shear forces. The computer program (CDBFDA) (Curved Deep Beam Finite Difference Analysis Program) coded in fortran-77 for the analysis of curved deep beams on elastic foundations was formed. The results from this method are compared with other methods exact and numerical and check the accuracy of the solutions. Good agreements are found, the average percentages of difference for deflections and moments are (5.3%), and (7.3%), respectively which indicate the efficiency of the adopted method for analysis.

Keywords: curved deep beam, finite differences, elastic foundations.

INTRODUCION

The object of the paper is to analyze curved deep beam using finite difference method. The beam is resting on elastic foundation with Winkler frictional and compressional resistances, and loaded generally (both transverse distributed load and distributed moment), and include the effect of transverse shearing deformations.

The linear elastic behavior of curved deep beams on elastic foundations is studied. The governing differential equations of curved deep beams (in terms of w and Ψ) are developed and converted into finite differences. A computer program in (FORTRAN language) is developed. This program assembling the finite difference equations to obtain a system of simultaneous algebraic equations and than solved by using Gauss elimination method. The deflections and rotations for each node are obtained. The shear and moment are obtained by simply substitutions of the deflections and rotations into the finite difference equations of moment and shear. The obtained solution compared with available results to check the accuracy of this method. Curved beams are one dimensional structural element that can sustain transverse loads by the development of bending, twisting and shearing resistances in the transverse sections of the beam. It's extensively used in engineering and other fields since such beams have many practical applications. The curved beam elements on elastic foundation would be helpful for the analysis of ring foundation of structures such as antennas, water towers structures, transmission towers and various other possible structures and superstructures. Deep beam model is based on the Timoshenko theory. This theory considers the effect of transverse shearing deformations. Thus, the cross sections of the beam remain plane but not normal to the axis of bending.

These are review of early studies on curved beams:

Volterra (1952) analyzed the deflections of circular beams resting on elastic foundations. The beam was loaded by symmetric concentrated forces acting in a plane perpendicular to the plane of the original curvature of the beam. The foundation is supposed to react following the classical Winkler and Zimmermann hypotheses, i.e., the reaction forces due to the foundation are proportional at every point to the deflection of the beam at that point.

Rodriguez, (1959), solved the three dimensional bending of a ring (curved beam in the form of a complete circle) of uniform cross sectional area and supported on a transverse elastic foundation.

Close, (1964), presented a mathematical analysis for determining the vertical deflection at the free end of a circular cantilever I-beam (curved in the horizontal plane). Chaudhuri and Shore, (1977), worked on thin walled curved beam finite elements, they presented a procedure for the consistent matrix formulation of a thin walled circularly curved beam element.

Yoo, (1979), presented matrix formulation for the static analysis of the spatially curved beams of thin walled members. This formulation was quite general and consistent.

Fukumoto and Nishida, (1981), derived a fundamental equations of a single curved I-beam subjected to the action of bending and torsional moments. They investigated the behavior of curved flexural members under large torsional deflection.

Dasgupta and Sengupta, (1988), suggested a formula for the analysis of a horizontal curved beam by using three node isoparametric finite elements. The formulation presented was general and the method, therefore, may be utilized for straight beams as well. The beam was with or without an elastic base throughout its length.

FORMULATION

According to the small deflection theory and linear stress-strain relationships, a formulation for the bending of curved deep beams is presented herein, and based on the following assumptions:

a) Plane cross-section before bending remains plane after bending.

b) The cross-section will have additional rotation due to transverse shear warping of the cross-section by transverse shear will be taken into consideration by introducing a shear correction factor (c^2).

Kinematics considerations

A cross section in θz plane is considered. The deflection w and the rotation of transverse section ψ in θ direction are shown in Figure-1.

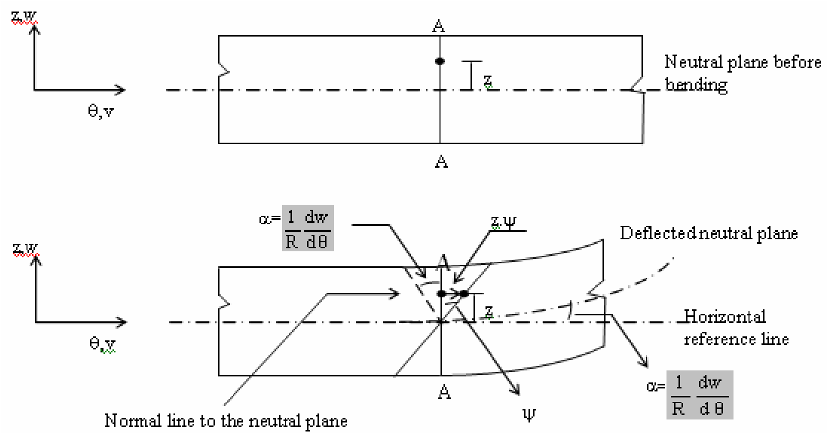

Figure-1. Deformations of a beam section.

A normal line to the neutral plane has two degrees of freedom (deflection w and rotationψ). The displacement in θ-direction (v) at a point at distance z above the neutral plane will be:

$$v = z.\ \psi \tag{1}$$

where

v = v(θ) is the displacement at neutral plane.

w = w(θ) (independent of z),

ψ = ψ(θ) (positive when in clockwise direction)

$$u = 0 \tag{2}$$

The mathematical expressions of strains are:

$$\varepsilon_\theta = \frac{1}{R}(\frac{dv}{d\theta}) + \frac{u}{R}$$

$$\varepsilon_\theta = \frac{z}{R}(\frac{d\psi}{d\theta}) \tag{3}$$

Also, the engineering shearing strains are:

$$\gamma_{\theta z} = (\frac{dv}{dz}) + \frac{1}{R}(\frac{dw}{d\theta})$$

$$\gamma_{\theta z} = \psi + \alpha \tag{4}$$

$$\alpha = \frac{1}{R}(\frac{dw}{d\theta}) \tag{5}$$

where α is the slope of the deflection curve.

The non-zero stresses are :

$$\sigma_\theta = E\varepsilon_\theta = E\frac{z}{R}(\frac{d\psi}{d\theta}) \tag{6}$$

$$\tau_{\theta z} = G\gamma_{\theta z} = G(\psi + \alpha) = G\left[\psi + \frac{1}{R}(\frac{dw}{d\theta})\right] \tag{7}$$

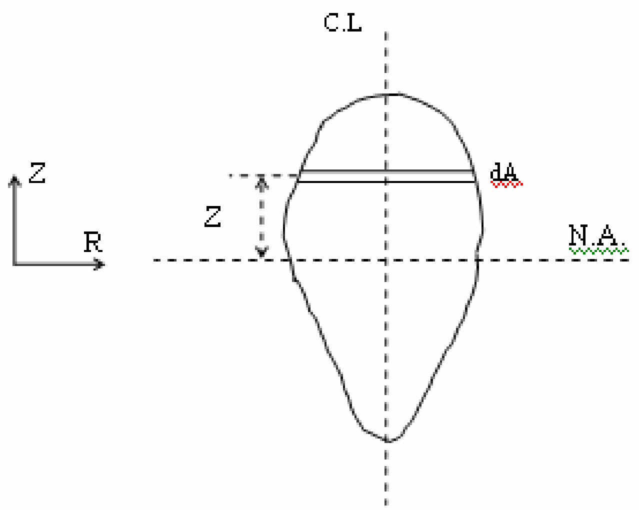

Figure-2. Arbitrary cross section of a curved deep beam.

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From Figure-2, the bending moment is:

$$M = \int_A \sigma_\theta \cdot z \cdot dA \qquad (8)$$

Using equation (6):

$$M = \int_A \frac{E}{R} z \frac{d\psi}{d\theta} z dA$$

$$M = \frac{EI}{R}\left(\frac{d\psi}{d\theta}\right)$$

(9)

Where $I = \int_A z^2 dA$ is the second moment of area of the cross-section. Also, using equation (7), the transverse shearing force is:

$$Q = Gc^2 A\left[\psi + \frac{1}{R}\left(\frac{dw}{d\theta}\right)\right] \qquad (10)$$

Where c^2 is numerical factor representing the restraint of the cross section against warping, commonly assumed to be (5/6) for rectangular sections.

Static considerations

Figure-3 shows the vertical and moment equilibrium.

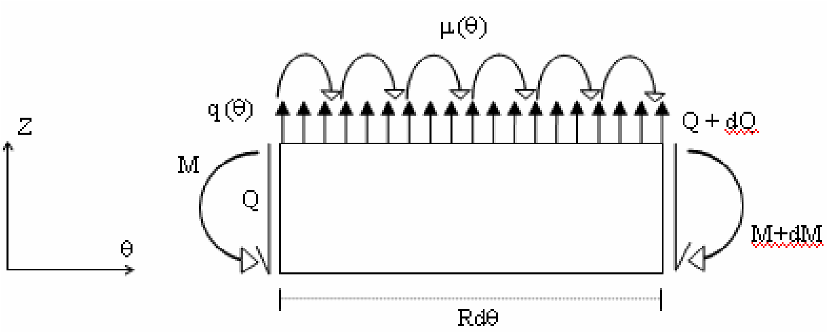

Figure-3. Curved beam under applied loadings.

By equilibrium of forces in z-direction:

$$Q + dQ - Q + q.\ R.\ d\theta = 0$$

or

$$\frac{dQ}{d\theta} = -q.R \qquad (11)$$

Equilibrium of moments in θZ- plane, gives:

$$M + dM - M - Q.R.\ d\theta + \mu.R.\ d\theta + q.\ \frac{(Rd\theta)^2}{2} = 0$$

The last term will be ignored (very small), thus

$$\frac{dM}{d\theta} = Q.R - \mu.R$$

(12)

where $\mu = \mu(\theta)$ is the distributed moment (per unit length).

By substituting equation (10) into equation (11), and both equations (9) and (10) into equation (12), then :

$$Gc^2A\left[\left(\frac{d\psi}{d\theta}\right) + \frac{1}{R}\left(\frac{d^2w}{d\theta^2}\right)\right] = -q.R \qquad (13)$$

$$\frac{EI}{R}\left(\frac{d^2\psi}{d\theta^2}\right) = Gc^2A.R\left[\psi + \frac{1}{R}\left(\frac{dw}{d\theta}\right)\right] - \mu.R \qquad (14)$$

Equations (13) and (14) are the governing differential equations of curved deep beams in terms of two deformation functions (w and ψ). These equations are coupled through the deformation functions.

Governing equations of curved deep beams on elastic foundations

The governing equations of curved deep beams on elastic foundations characterized by Winkler model for both compressional and frictional resistances are given:

$$Gc^2A\left[\frac{d\psi}{d\theta} + \frac{1}{R}\frac{d^2w}{d\theta^2}\right] + qR - K_z w.R = 0 \qquad (15)$$

$$\frac{EI}{R}\left(\frac{d^2\psi}{d\theta^2}\right) - Gc^2A.R\left[\psi + \frac{1}{R}\frac{dw}{d\theta}\right] - K_\theta\left(\frac{h}{2}\right)^2 \psi.R = 0 \qquad (16)$$

FINITE DIFFERENCE ANALYSIS

The governing differential equations for curved deep beams on elastic foundations represented by a Winkler model for frictional restraints can be rewritten using finite differences equations for an interior node (i):

$$Gc^2A\left[\frac{\psi_{i+1} - \psi_{i-1}}{2\Delta\theta}\right] + \frac{Gc^2A}{R\,\Delta\theta^2}\left[w_{i+1} - 2w_i - \frac{K_z w_i R^2 \Delta\theta^2}{Gc^2A} + w_{i-1}\right] + q_i R = 0 \qquad (17)$$

$$\frac{EI}{R}\left[\frac{\psi_{i+1}-2\psi_i+\psi_{i-1}}{\Delta\theta^2}\right]-Gc^2AR[\psi_i+\frac{1}{R}(\frac{w_{i+1}-w_{i-1}}{2\Delta\theta})-K_\theta(\frac{h}{2})^2\psi_i.R=0 \quad (18)$$

APPLICATIONS

Ring beam on Winkler foundation under four concentrated loads

A ring foundation of (E = 20.7 kN/mm^2, ν = 0.15), having a radius of (R = 7629.45 mm), width (b = 762 mm), thickness (h = 762mm). The ring beam carries four equal column loads (perpendicular to the ring), each column load (P = 667.5 kN). The ring beam is resting on an elastic foundation which is represented by Winkler model for compressional restraint with the coefficient (K_Z = 0.135*10^{-4} kN/mm^3), as shown in Figure-4. This problem was solved by Dasgupta and Sengupta, (1988). In the present study, the same problem is solved by using the finite difference method. The results of deflections and bending moments are plotted with the results of Dasgupta and Sengupta, (1988) as shown in Figures 5 and 6.

The figures show acceptable agreement. The percentage of the difference between the maximum deflections and bending moments for Dasgupta and Sengupta,(1988) solution and the present study is equal to (4.3%), (1.48%), respectively.

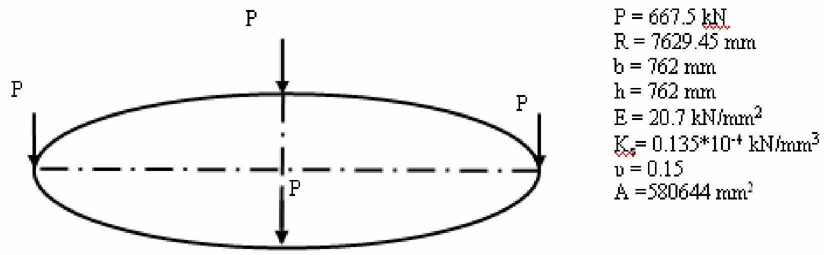

Figure-4. Ring beam on an elastic foundation under four concentrated loads.

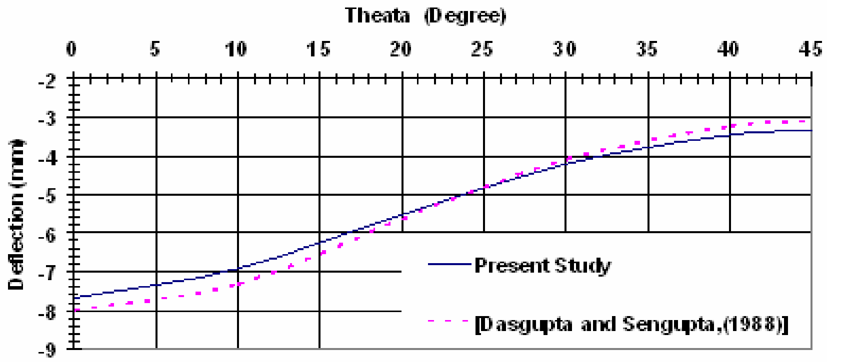

Figure-5. Deflection curves for free curved beam resting on an elastic foundation.

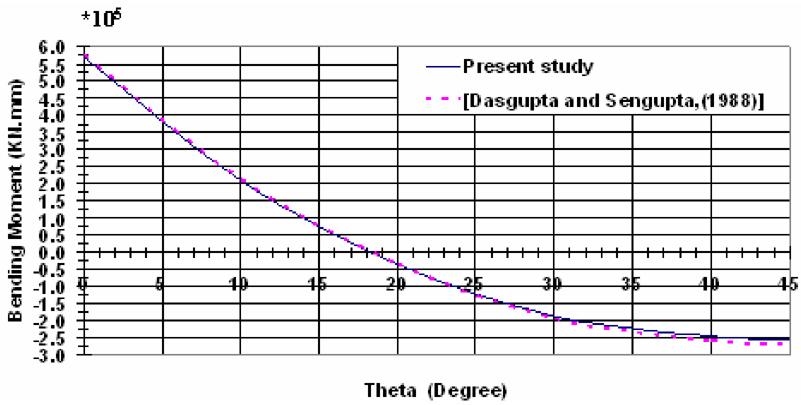

Figure-6. Bending moment diagrams for free curved beam resting on an elastic foundation.

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Fixed ends curved deep beam resting on an elastic foundation under uniform loading

A fixed curved deep beam with Young's modulus of (E = 25 kN/mm^2), Poisson's ratio (ν = 0.15), radius (R = 1000 mm), width (b = 100mm), and thickness (h = 400 mm), and loaded by a uniform load (q = 0.025 kN/mm) is considered. The load is acting in a plane perpendicular to the plane of the original curvature of the beam and at a total angular distance (θ = 180°). The beam is resting on an elastic foundation which is represented by Winkler model for compressional restraint with coefficient (K_Z = 0.1*10^{-4} kN/mm^3) and by Winkler friction model with spring coefficient (K_θ = 0.2*10^{-4} kN/mm^3), as shown in Figure-7. This problem was solved by using Plate Foundation Analysis Program (PFAP) [Al-Allaf, (2005)]. In the present study, the problem is solved by using the finite difference method. The results of deflections obtained from present study are plotted with the results of (PFAP) as shown in Figure-8. The Figure shows acceptable agreement. The percentage of the difference between the maximum deflections for (PFAP) solution and the present study is (9.4%).

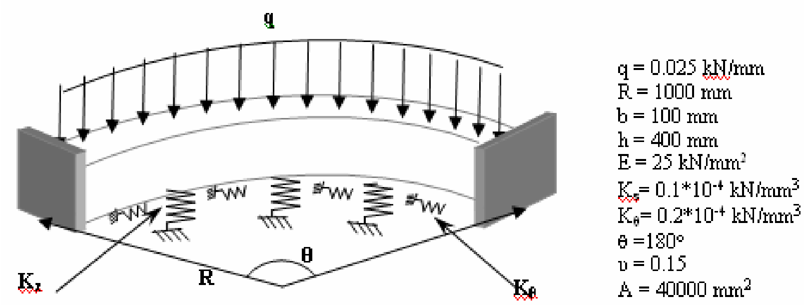

Figure-7. Fixed ends curved deep beam resting on an elastic foundation Under uniform distributed load (q).

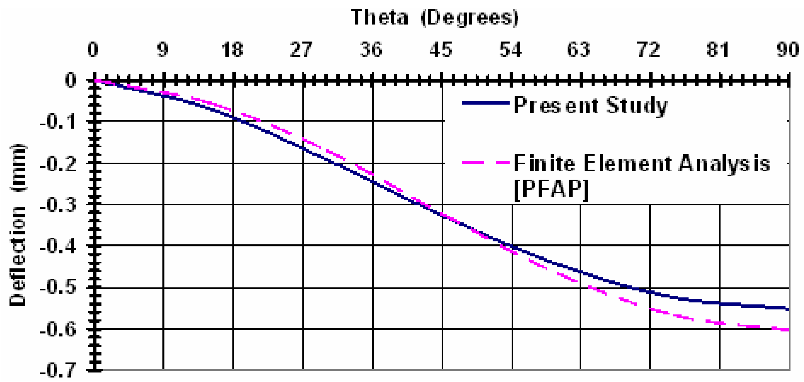

Figure-8. Deflection curves for fixed ends curved deep beam resting on an elastic foundation.

PARAMETRIC STUDY

A parametric study is performed to show the influence of several important parameters on the behavior of free curved beam. The values of the vertical and horizontal subgrade reactions (0.13577*10^{-3} kN/mm^3 and 0.2*10^{-3} kN/mm^3) are considered.

The effect of increasing the thickness to radius ratio (h/R) is shown in Figure-9. From this Figure the maximum deflection will decrease at decreasing rate as the beam thickness increased. It was found that by increasing the ratio of (h/R) from (0.1 to 1), the maximum deflection for the free curved beam under four concentrated loads are decreased by (63.522%).

The effect of increasing the beam width is shown in Figure-10. From this Figure the maximum deflection will decrease at decreasing rate as the beam width increased. It was found that by increasing the width of the free curved beam from (762 to 3048 mm), the maximum deflection for the curved beam under four concentrated loads are decreased by (75%).

The effect of increasing the vertical and horizontal subgrade reactions (K_z) and (K_θ) are shown in Figures 11 and 12. From these Figures, the maximum deflection will decrease at a decreasing rate as the vertical subgrade reaction is increased. While the maximum deflection will decrease almost linearly as the horizontal subgrade reactions is increased. It was found that by increasing the vertical and horizontal subgrade reactions for the free curved beam from (0.135*10^{-3} kN/mm^3 to 0.6*10^{-3} kN/mm^3) for the vertical, and from (0.2*10^{-3} kN/mm^3 to 0.1*10^{-2} kN/mm^3) for the horizontal, the maximum deflection decreased by (65.3%), (7.184%), respectively.

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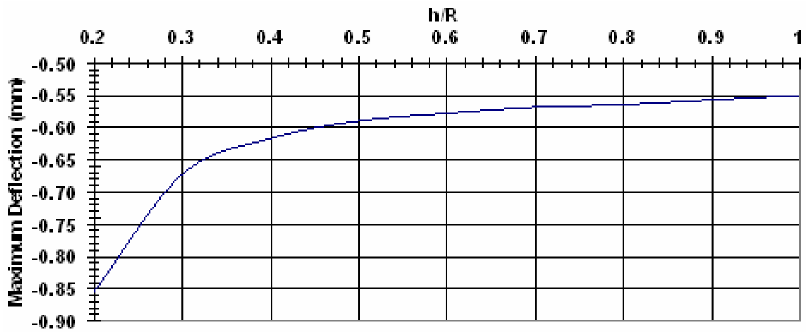

Figure-9. Effect of (h/R) on the maximum deflection for free curved beam under four concentrated loads.

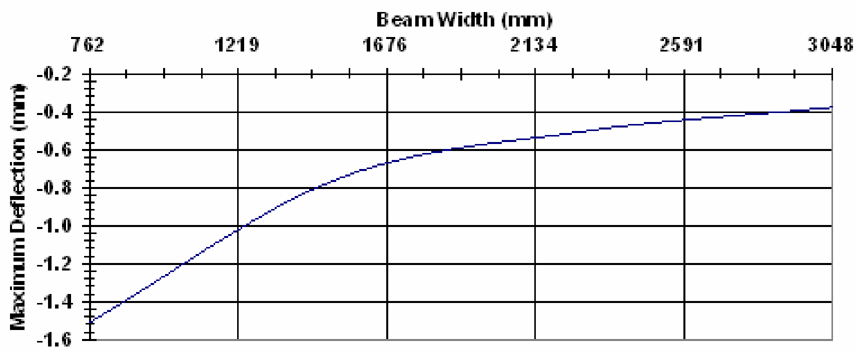

Figure-10. Effect of beam width on the maximum deflection for free curved beam under four concentrated loads.

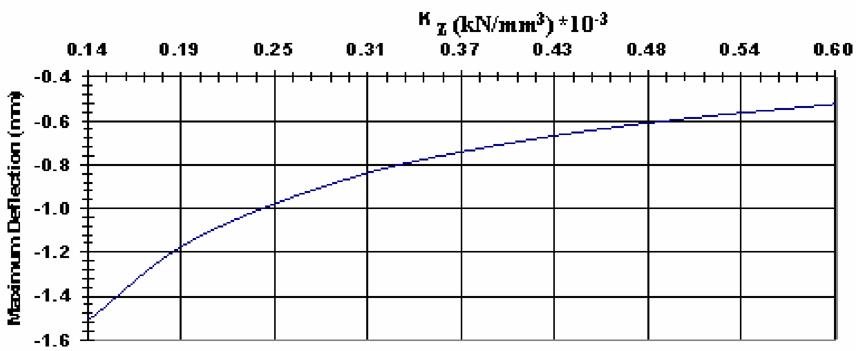

Figure-11. Effect of vertical subgrade reaction on maximum deflection for free curved beam under four concentrated loads.

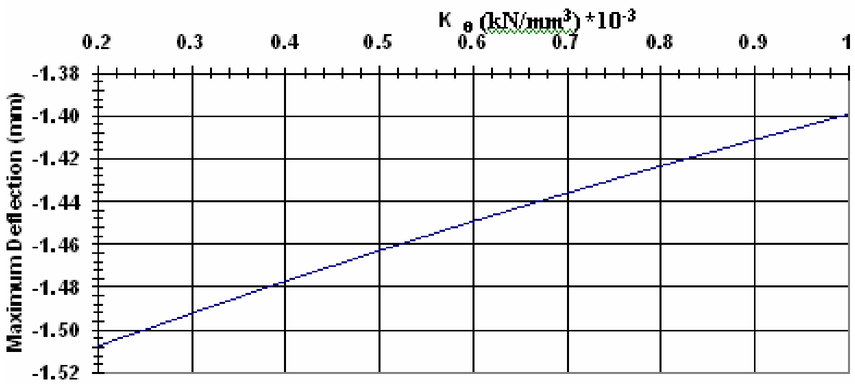

Figure-12. Effect of horizontal subgrade reaction on maximum deflection for free curved beam under four concentrated loads.

CONCLUSIONS

a) Original Timoshenko's deep beam theories are extended to include the effects of externally distributed moments along the curved deep beam and modeled by finite difference;
b) The benefit of including the externally distributed moments to the original theories for beams is to increase the domain of applications. The distributed moments can be generated from various problems; externally applied distributed moments, the problems of curved deep beams on elastic foundations with frictional restraints, curved deep beam subjected to temperature gradients, prestressed curved deep beams and the problem of shrinkage in concrete in reinforced concrete curved deep beams;
c) In the problems of curved deep beams on elastic foundations, various models can be suggested to represent both the compressional and frictional restraints. The frictional component can be represented by Winkler model (proportional to the horizontal displacements) or by Coulomb model (proportional to the transverse displacements) or by a constant value (independent of horizontal and transverse displacements);
d) The numerical techniques of the finite-difference method are used to solve the problem of curved deep beams resting on elastic foundations. The formulations in finite-difference method are based on the governing equations; and
e) When the width of the beam increases, the deflection will decrease.

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