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APPROXIMATE METHOD TO SINGULARITY BEHAVIOUR OF NONLINEAR PROBLEMS IN FLUID DYNAMICS

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ABSTRACT

The singularity behaviour of nonlinear problems in fluid dynamics is numerically investigated in the present paper. The behaviour of the first Painlev'e transcendent is introduced as a model problem. Then the laminar unsteady flow of a viscous fluid away from a plane stagnation point is studied as second model problem. We have applied approximate method to these problems with the aid of algebraic programming language MAPLE. Analysis based on approximate method suggests that the convergence of the series of shear stress is limited by a pair of singularities. The location and nature of the singularities in the complex plane are presented. The shear stress in real plane is also depicted graphically.

Keywords: fluid dynamics, singularity, approximate method, unsteady, shear stress, stagnation point.

Nomenclature

U_0	Speed of the stream at infinity
ρ	Density of the fluid
ψ	Stream function
a	Characteristic length
ν	Kinematics viscosity
t'	Time
α	Critical exponent
τ'	Shear stress
HODA	High Order Differential Approximant
D-T	Drazin- Tourigny Approximant
θ_c	Critical values of angle with the positive real axis
r_c	Critical values of radius of convergence
t_c	Critical values of time

1. INTRODUCTION

Model of physical phenomena is most of the time represented by nonlinear differential equations. Practically, it is difficult to determine a closed form of the exact solution. But one can extract a series solution. Still presence of singularity prevents rapid convergence of the series. So it is necessary to seek an efficient approximate method. Approximate methods are the techniques for summing power series. A function is said to be approximant for a given series if its Taylor series expansion reproduces the first few terms of the series. $Hermite-Pade'$ Class introduced semi numerical approximate methods, such as Pade' approximants, algebraic and differential approximants etc. More recently Drazin and Tourigny [1] made some improvement to the algebraic approximant where they extended the theory for large N. The detailed information about the Drazin - Tourigny method will be found in [1]. Khan [2] introduced High-order differential approximant in his PhD thesis which is an extension of differential approximant. A comparative performance of High-order differential approximant and Drazin-Tourigny approximant to determine the singularity of two nonlinear model problems in fluid dynamics are analyzed in the paper.

In the present study, we introduce the singularity behaviour of the solution of a nonlinear differential equation from Bender and Orszag [3] namely the first Painlev'e transcendent as model problem 1. Bender and Orszag [3] discussed a number of examples with local analysis. Without solving the equation they tried to locate the dominating singular points of this kind of nonlinear differential equations by the application of approximate method. Bender and Orszag [3] discovered by asymptotic analysis that the solution of the first Painlev'e transcendent has an infinite number of second-order poles along the real axis and the nearest pole to the origin is approximately 3.7428. In our study we investigate the singular point with nature of the first Painlev'e transcendent applying approximate methods.

Finally, the singularity behaviour of the laminar unsteady flow of an incompressible fluid with small viscosity away from a stagnation point is presented as model problem 2.

The two-dimensional flow in the neighborhood of the rear stagnation point on a cylinder which is set in motion impulsively with constant velocity was first analyzed by Proudman and Johnson [4]. The asymptotic expansion of the flow for large time outside the viscous layer on the boundary was obtained by them. Robins and Howarth [5] extended the asymptotic expansion to higher order and calculated the numerical solution that supported the analytical solution. However, the exact solution remained undetermined.

Hommel [2] investigated the laminar flow of an incompressible fluid with small viscosity away from a stagnation point. The non-dimensional equation was obtained from the Navier-Stokes equation using similarity

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analysis. Then Hommel [2] solved this equation as a series with 44 terms in time for the shear stress at the stagnation point. Hommel [2] found that the convergence of this series is limited by a pole in the complex plane nearest to the origin. He showed that the radius of convergence of the series is approximately 3 according to Cauchy root test and following Van Dyke [6] sign pattern analysis, confirms the existence of a complex conjugate pair of singularities forming an angle $\theta = \pm 67.5$ with the real axis in the complex plane. He introduced a linear fractional transformation to banish the pole to infinity to improve the convergence of the series for infinite time.

In this paper we illustrate the location and nature of the singular point of the series of shear stress applying approximate methods with the help of algebraic programming language MAPLE. In our analysis, High-order differential approximant method determines the pair of singular points of time with location and nature accurately compared to Hommel [2]. The graph of the shear stress for infinite time, an extension of Hommel [2], is also produced by Drazin-Tourigny method.

2. REVIEW OF PADE′-HERMITE PPROXIMANTS

In 1893, Pade′ and Hermite introduced Pade′-Hermite class. All the one variable approximants that were used or discussed throughout this paper belong to the Pade′-Hermite class. In its most general form, this class is concerned with the simultaneous approximation of several independent series.

Let $d \in \mathrm{N}$ and let the $d+1$ power series $U_0(x), U_1(x), ..., U_d(x)$ are given.

Assume that the $(d+1)$ tuple of polynomials $P_N^{[0]}, P_N^{[1]}, ..., P_N^{[d]}$, where

$$\deg P_N^{[0]} + \deg P_N^{[1]} + ... + \deg P_N^{[d]} + d = N, \tag{1}$$

is a Pade′-Hermite form of these series if

$$\sum_{i=0}^{d} P_N^{[i]}(x) U_i(x) = O(x^N) \text{ as } x \to 0. \tag{2}$$

Here $U_0(x), U_1(x), ..., U_d(x)$ may be independent series or different form of a unique series. We need to find the polynomials $P_N^{[i]}$ that satisfy the equations (1) and (2). These polynomials are completely determined by their coefficients. So, the total number of unknowns in equation (2) is

$$\sum_{i=0}^{d} \deg P_N^{[i]} + d + 1 = N + 1 \tag{3}$$

Expanding the left hand side of equation (2) in powers of *x* and equating the first *N* equations of the system equal to zero, we get a system of linear homogeneous equations. To calculate the coefficients of the Pade′-Hermite polynomials it require some sort of normalization, such as $P_N^{[i]}(0) = 1$ for some $0 \le i \le d$

It is important to emphasize that the only input required for the calculation of the Pade′-Hermite polynomials are the first *N* coefficients of the series $U_0, ..., U_d$. The equation (3) simply ensures that the coefficient matrix associated with the system is square. One way to construct the Pade′-Hermite polynomials is to solve the system of linear equations by any standard method such as Gaussian elimination or Gauss-Jordan elimination.

Drazin- Tourigney Approximants (2003) is a particular kind of algebraic approximants and Khan (2002) introduced High-order differential approximant as a special type of differential approximants. An algebraic programming language Maple available on www.maplesoft.com was used to compute the series coefficients of non-dimensional governing equation of the problem.

3. MODEL PROBLEM 1

The first Painlev′e Transcendent as $\mathrm{X} \to +\infty$:

The following nonlinear differential equation with initial conditions is considered

$$u'' = u^2 + x, u(0) = u'(0) = 0 \tag{4}$$

This differential equation is the first of a set of six equations whose solutions are called the Painlev'e transcendent. These equations were discovered by Painlev'e in the course of classifying nonlinear differential equations. He considered all equations of the form $w'' = R(z,w)(w')^2 + S(z,w)w' + T(z,w)$ having the properties (a) that R, S and T, are rational functions of w, but have arbitrary dependence on z and (b) that the solutions (1.2) may have various kinds of fixed singularities (poles, branch points, essential singularities), but may not have any movable singularities except for poles. By asymptotic analysis Bender and Orszag [3] indicate that $u(x)$ has an infinite number of second-order poles along the positive real axis and the nearest pole to the origin is $x_{c,N} \approx 3.7428$. They have plotted the result in figure to support that there is a sequence of poles along the positive real axis.

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Table 1. Estimates of the critical point $x_{c,N}$ and the corresponding exponent α by using High-order differential approximants [7] (HODA) and Drazin and Tourigney [1] (D-T) method.

N	d	HODA $x_{c,N}$	HODA α	N	d	D-T $x_{c,N}$
12	3	3.7428014612217258443	-1.9999967546910651			
18	4	3.7428015307707560902	-2.00000000000131121			
25	5	3.742801530770741877	-2.00000000000000000			
33	6	3.7428015307707418775	-1.9999999999999999	25	6	3.7421673901815414712
42	7	3.7428015307707418775	-1.99999999999999999	33	7	3.7426267904015566872
52	8	3.7428015307707418778	-2.00000000000000002	42	8	3.7404447944582245918
63	9	3.7428015307752701046	-2.00000000015497759	52	9	3.7074398382066125923
75	10	3.7428015307494165916	-1.99999999532936381			

The singularity and its nature of the nonlinear differential equation (4) are analyzed by approximate methods in the present study. Therefore, the results in Table-1 display the convergence of $x_{c,N}$ up to 19 decimal places at $d = 7$ taking $N = 42$ coefficients of the series solution $u(x)$ of (4). It can be also noted that the values of α confirm that $x_{c,N}$ is pole by using the High-order differential approximants (HODA). For comparison, Table-1 also shows the results by the Drazin-Tourigny (D-T) approximants. It is clear that the High-order differential approximant converges much faster and the value of α confirm that $x_{c,N}$ is a pole.

We now study the laminar unsteady flow of a viscous fluid at a plane stagnation point, where the singularity, for which the convergence of the series is limited, lies in the complex plane is a pole. Approximate methods are applied to determine this singularity in the complex plane and its critical exponent.

4. MODEL PROBLEM 2

The laminar unsteady flow of a viscous fluid away from a plane stagnation point is considered. The non-dimensional coordinates are considered in Hommel [2] as,

$$x = \frac{x'}{a}, \qquad y = y'\left(\frac{2U_0}{\nu a}\right)^{1/2}, \qquad t = \frac{2U_0 t'}{a}$$

where x' and y' are dimensional coordinates defined tangential and normal to the flow boundary respectively, with x' measured away from the stagnation point. The stream function of the potential flow in a small neighborhood near the stagnation point due to an impulsive start is defined by $\psi = -(2\nu aU_0)^{1/2} xy$, imposed by Proudman and Johnson [4], as the outer boundary condition for all time.

Since the flow is time-dependent, $y = F(y,t)$ is considered and the stream function is transformed as $\tilde{\psi} = (2\nu aU_0)^{1/2} xF(y,t)$ to solve the Navier-Stokes equations exactly.

The following non-dimensional governing equation with initial and boundary conditions are obtained by solving the Navier-Stokes equations

$$\begin{aligned} &F_{yt} - F_{yyy} = \left(-1 + F_y^2 - FF_{yy}\right), \\ &F(0,t) = F_y(0,t) = 0, F_y(\infty,t) = 1, \\ &F_y(y,0) = 1(y \neq 0). \end{aligned} \tag{5}$$

To solve the non-dimensional governing equation (5), a power series is considered in the form

$$F(\eta) = 2\sqrt{t}\sum_{i=1}^{\infty} f_i(\eta)t^{i-1} \tag{6}$$

Where, $\eta = y/2t^{1/2}$

Substituting (6) into the equation (5) and using the finite difference scheme the 44 terms of the dimensional shear stress

$$\tau'(x,y) = \rho\nu^{1/2}\left(\frac{U_0}{a}\right)^{3/2} x' \frac{1}{2t^{1/2}} \sum f_n''(0)t^{n-1} \tag{7}$$

at the boundary is computed by Hommel [2]. Then in the present work differential and algebraic approximate methods are applied into the series (7) to determine the location and nature of the singular point t_c in the complex plane. Also the shear stress for infinite time is shown graphically using Drazin- Tourigny approximate method.

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5. RESULTS AND DISCUSSIONS

High-order Differential and Drazin - Tourigny approximate methods are applied to the coefficient of the series (7) to determine the location of the singular point with its nature numerically. Table-2 represents the complex conjugate pair of singular points

$t_c \approx 1.176206370 \pm 2.759056686i$ calculated by High-order Differential Approximant [7] at $d = 7$ taking $N = 42$ coefficients of the series (7). It is observed from Table-2 that both the High-order Differential and Drazin-Tourigny approximate methods computed approximately the same value of $r_c \approx 3$. Moreover, in Table-2 High-order differential approximate calculated the critical angle $\theta_c = \pm 67.04$ at $d = 7$ taking $N = 42$. Therefore, both the values r_c and θ_c found in Table-2 are in good agreement with Hommel [2]. Furthermore, the value of α at $d = 7$ in Table-2 confirms that t_c is a pole sustains the analysis of Hommel [2] about the nature of the singularities.

Table 2. Calculated values of singularity t_c, critical radius of convergence r_c, critical angle with the positive real axis θ_c and the critical exponent α using Drazin-Tourigny method [1] and High-order differential approximants [7].

D-T				
d, N	t_c	r_c	θ_c	
5,20	$.831095912630 + 3.0883573518i$	3.19822944	74.93	
6,27	$2.93037916321 + 3.4179004117214i$	4.50212897	49.38	
7,35	$1.250590891999 - 2.7191540106916i$	2.99295441	- 65.30	
HODA				
d, N	t_c	r_c	θ_c	α
5, 25	$1.0988974417 \pm 2.3950727668i$	2.63513739	± 65.47	$-39.762218 + 22.1007012i$
6,33	$1.377750991 \pm 3.017342269i$	3.31700952	± 65.57	$10.2145653 + 4.62881593i$
7,42	$1.1762063770 \pm 2.7590566866i$	2.999309125	± 67.04	$-2.070123085 + 6.106652415i$

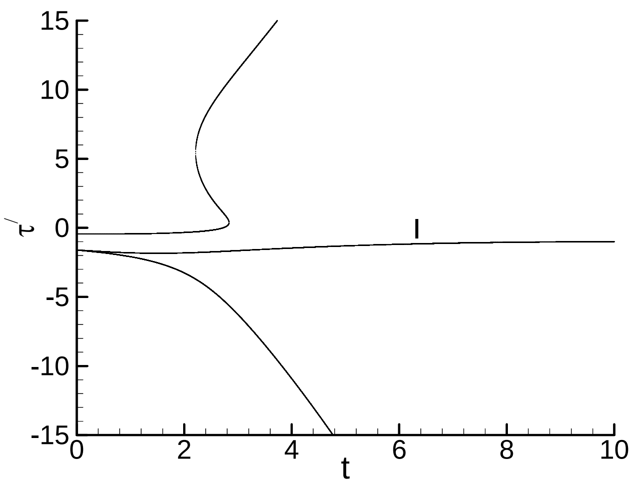

Figure 1. Approximate solution diagram (curve I) in the (t, τ') plane obtained by Drazin-Tourigny method [1] for $d = 6$. The other curves are spurious.

The Drain-Tourigny method [1] is applied to the series coefficients of (7) in order to determine the shear stress τ'. Figure 1 displays the shear stress τ' approaches -1.5 for all time at $d = 6$, whereas the shear stress τ' approaches -1 if the degree increased to $d = 7$.

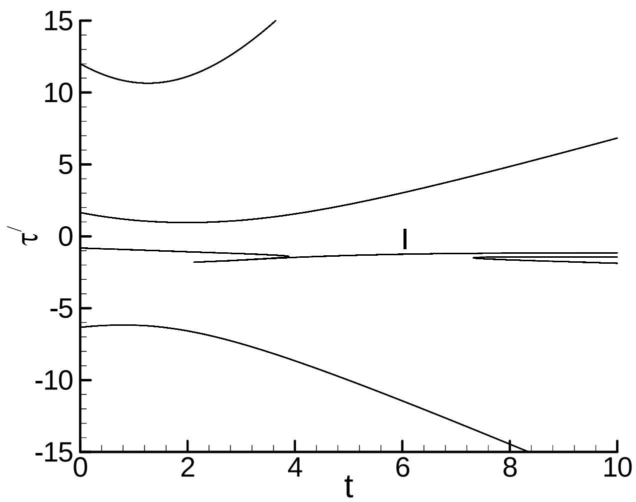

Figure 2. Approximate solution diagram (curve I) in the (t, τ') plane obtained by Drazin-Tourigny method [1] for $d = 7$. The other curves are spurious.

However, Hommel [2] calculation shows that $\tau' \approx -1.23259$. Finally, it is observed that the

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conjecture of the figures support the results of Hommel [2].

CONCLUSIONS

As model problem one the first Painlev'e transcendent is exhibited in order to investigate the singularities by approximate methods. The singular point found nearest to the origin is a pole converges to 3.7428. The singularity behaviour of the laminar unsteady flow of a viscous fluid away from a plane stagnation point is studied as model problem two. The series of shear stress of the non-dimensional governing equation with initial and boundary conditions is analyzed by approximate method. The singularities calculated are complex poles lay at a radius of convergence approximately 3 and making angle ± 67.04 with real axis. The value of shear stress shown graphically approximately -1 for all time. The work is comparable with Hommel [2], but our result is more accurate using less number of series coefficients. We believe that it can be improved by using more series coefficients. Approximate method gives a proper guidance to reveal the significant features of the singularities.

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