A MATHEMATICAL MODEL OF FOUR SPECIES SYN-ECOSYMBIOSIS COMPRISING OF PREY-PREDATION, MUTUALISM AND COMMENSALISMS-I (FULLY WASHED OUT STATE)

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ABSTRACT

This investigation deals with a mathematical model of a four species (S_1, S_2, S_3 and S_4) Syn-Ecological system (Fully Washed out State). S_2 is a predator surviving on the prey S_1: the prey is a commensal to the host S_3 which itself is in mutualism with the fourth species S_4. S_2 and S_4 are neutral. The mathematical model equations characterizing the syn-ecosystem constitute a set of four first order non-linear coupled differential equations. There are in all sixteen equilibrium points. Criteria for the asymptotic stability of one of the sixteen equilibrium points: the fully washed out state is established. The linearised equations for the perturbations over the equilibrium point are analyzed to establish the criteria for stability. The system is noticed to be locally stable. Trajectories of the perturbations have been illustrated.

Keywords: mathematical model, species, syn-ecological system, mutualism, commensalism, differential equations.

INTRODUCTION

Mathematical modeling is an important interdisciplinary activity which involves the study of some aspects of diverse disciplines. Biology, Epidemiodology, Physiology, Ecology, Immunology, Bio-economics, Genetics, Pharmocokinetics are some of those disciplines. This mathematical modeling has raised to the zenith in recent years and spread to all branches of life and drew the attention of every one. Mathematical modeling of ecosystems was initiated by Lotka [8] and by Volterra [14]. The general concept of modeling has been presented in the treatises of Meyer [9], Cushing [2], Paul Colinvaux [10], Freedman [3], Kapur [5, 6]. The ecological interactions can be broadly classified as prey-predation, competition, mutualism and so on. N.C. Srinivas [13] studied the competitive eco-systems of two species and three species with regard to limited and unlimited resources. Later, Lakshmi Narayan [7] has investigated the two species prey-predator models. Recently stability analysis of competitive species was investigated by Archana Reddy [1]. Local stability analysis for a two-species ecological mutualism model has been investigated by B. Ravindra Reddy *et al.*, [11, 12].

The present investigation is devoted to an analytical study of a four species Syn-Ecological system. S_2 is a predator surviving on the prey S_1: the prey is a commensal to the host S_3 which itself is in mutualism with the fourth species S_4; S_2 and S_4 are neutral. Figure-1 shows the Schematic Sketch of the system under investigation. The model equations of the system constitute a set of four first order non-linear ordinary differential coupled equations. In all the sixteen equilibrium points of the system are identified and the stability analysis is carried out only for the fully washed out state. The linearized perturbed equations over the equilibrium states are solved and the trajectories illustrated.

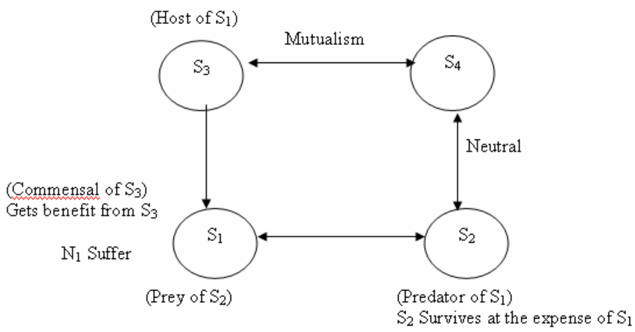

Figure-1. Schematic sketch of the Syn Eco- system.

BASIC EQUATIONS

Notation adopted

N_1 (t): The Population of the Prey (S_1)
N_2 (t): The Population of the Predator (S_2)
N_3 (t): The Population of the Host (S_3) of the Prey (S_1) and mutual to S_4
N_4 (t): The Population of S_4 mutual to S_3
t: Time instant
a_1, a_2, a_3, a_4: Natural growth rates of S_1, S_2, S_3, S_4
a_{11}, a_{22}, a_{33}, a_{44}: Self inhibition coefficients of S_1, S_2, S_3, S_4
a_{12}, a_{21}: Interaction (Prey-Predator) coefficients of S_1 due to S_2 and S_2 due to S_1
a_{13}: Coefficient for commensal for S_1 due to the Host S_3
a_{34}, a_{43}: Mutually interaction between S_3 and S_4

$\frac{a_1}{a_{11}}, \frac{a_2}{a_{22}}, \frac{a_3}{a_{33}}, \frac{a_4}{a_{44}}$: Carrying capacities of S_1, S_2, S_3, S_4

Further the variables N_1, N_2, N_3, N_4 are non-negative and the model parameters a_1, a_2, a_3, a_4; a_{11}, a_{22}, a_{33}, a_{44}; a_{12}, a_{21}, a_{13}, a_{24} are assumed to be non-negative constants.

The model equations for the growth rates of S_1, S_2, S_3, S_4 are

$$\frac{dN_1}{dt} = a_1N_1 - a_{11}N_1^2 - a_{12}N_1N_2 + a_{13}N_1N_3 \quad \text{....} \quad (1)$$

$$\frac{dN_2}{dt} = a_2N_2 - a_{22}N_2^2 + a_{21}N_2N_1 \quad \text{....} \quad (2)$$

$$\frac{dN_3}{dt} = a_3N_3 - a_{33}N_3^2 + a_{34}N_3N_4 \quad \text{....} \quad (3)$$

$$\frac{dN_4}{dt} = a_4N_4 - a_{44}N_4^2 + a_{43}N_4N_3 \quad \text{....} \quad (4)$$

EQUILIBRIUM STATES

The system under investigation has sixteen equilibrium states defined by

$$\frac{dN_i}{dt} = 0,\ i = 1, 2, 3, 4 \quad \text{......} \quad (5)$$

are given in the following table.

S. No.	Equilibrium States	Equilibrium point
1	Fully Washed out state	$\overline{N_1}=0, \overline{N_2}=0, \overline{N_3}=0, \overline{N_4}=0$
2	Only S_4 survives	$\overline{N_1}=0, \overline{N_2}=0, \overline{N_3}=0, \overline{N_4}=\frac{a_4}{a_{44}}$
3	Only the host (S_3)of S_1 survives	$\overline{N_1}=0, \overline{N_2}=0, \overline{N_3}=\frac{a_3}{a_{33}}, \overline{N_4}=0$
4	Only the predator S_2 survives	$\overline{N_1}=0, \overline{N_2}=\frac{a_2}{a_{22}}, \overline{N_3}=0, \overline{N_4}=0$
5	Only the prey S_1 survives	$\overline{N_1}=\frac{a_1}{a_{11}}, \overline{N_2}=0, \overline{N_3}=0, \overline{N_4}=0$
6	Prey (S_1) and predator (S_2) washed out	$\overline{N_1}=0, \overline{N_2}=0, \overline{N_3}=\frac{a_4a_{34}+a_3a_{44}}{a_{33}a_{44}-a_{34}a_{43}}, \overline{N_4}=\frac{a_3a_{43}+a_4a_{33}}{a_{33}a_{44}-a_{34}a_{43}}$
7	Prey (S_1) and host (S_3) of S_1 washed out	$\overline{N_1}=0, \overline{N_2}=\frac{a_2}{a_{22}}, \overline{N_3}=0, \overline{N_4}=\frac{a_4}{a_{44}}$
8	Prey (S_1) and S_4 washed out	$\overline{N_1}=0, \overline{N_2}=\frac{a_2}{a_{22}}, \overline{N_3}=\frac{a_3}{a_{33}}, \overline{N_4}=0$
9	Predator (S_2) and Host (S_3) of S_1 washed out	$\overline{N_1}=\frac{a_1}{a_{11}}, \overline{N_2}=0, \overline{N_3}=0, \overline{N_4}=\frac{a_4}{a_{44}}$
10	Predator (S_2) and S_4 washed out	$\overline{N_1}=\frac{a_1a_{33}+a_3a_{13}}{a_{11}a_{33}}, \overline{N_2}=0, \overline{N_3}=\frac{a_3}{a_{33}}, \overline{N_4}=0$
11	Prey (S_1) and predator (S_2)survives	$\overline{N_1}=\frac{a_1a_{22}-a_2a_{12}}{a_{11}a_{22}+a_{12}a_{21}}, \overline{N_2}=\frac{a_1a_{21}+a_2a_{11}}{a_{11}a_{22}+a_{12}a_{21}}, \overline{N_3}=0, \overline{N_4}=0$
12	Only the prey (S_1) washed out	$\overline{N_1}=0, \overline{N_2}=\frac{a_2}{a_{22}}, \overline{N_3}=\frac{a_4a_{34}+a_3a_{44}}{a_{33}a_{44}-a_{34}a_{43}}, \overline{N_4}=\frac{a_4a_{33}+a_3a_{43}}{a_{33}a_{44}-a_{34}a_{43}}$
13	Only the predator (S_2) washed out	$\overline{N_1}=\frac{\alpha_1}{\alpha_2}, \overline{N_2}=0, \overline{N_3}=\frac{a_4a_{34}+a_3a_{44}}{a_{33}a_{44}-a_{34}a_{43}}, \overline{N_4}=\frac{a_4a_{33}+a_3a_{43}}{a_{33}a_{44}-a_{34}a_{43}}$ *where* $\alpha_1 = a_{13}(a_4a_{34}+a_3a_{44})+a_1(a_{33}a_{44}-a_{34}a_{43})$ $\alpha_2 = a_{11}(a_{33}a_{44}-a_{34}a_{43})$

14	Only the Host (S_3) of S_1 washed out	$\overline{N_1} = \frac{a_1 a_{22} - a_2 a_{12}}{a_{11}a_{22} + a_{12}a_{21}}, \overline{N_2} = \frac{a_1 a_{21} + a_2 a_{11}}{a_{11}a_{22} + a_{12}a_{21}}, \overline{N_3} = 0, \overline{N_4} = \frac{a_4}{a_{44}}$
15	Only S_4 washed out	$\overline{N_1} = \frac{\beta_2}{\beta_1}, \overline{N_2} = \frac{\beta_3}{\beta_1}, \overline{N_3} = \frac{a_3}{a_{33}}, \overline{N_4} = 0$ *where* $\beta_1 = a_{33}(a_{11}a_{22} + a_{12}a_{21})$ $\beta_2 = a_{22}(a_1 a_{33} + a_3 a_{13}) - a_2 a_{12} a_{33}$ $\beta_3 = a_{21}(a_1 a_{33} + a_3 a_{13}) + a_2 a_{11} a_{33}$
16	The co-existent state (or) Normal steady state	$\overline{N_1} = \frac{\gamma_1 + a_{13}a_{22}\gamma_2}{\gamma_3}, \overline{N_2} = \frac{\gamma_4 + a_{13}a_{21}\gamma_2}{\gamma_3},$ $\overline{N_3} = \frac{a_4 a_{34} + a_3 a_{44}}{a_{33}a_{44} - a_{34}a_{43}}, \overline{N_4} = \frac{a_4 a_{33} + a_3 a_{43}}{a_{33}a_{44} - a_{34}a_{43}}$ *where* $\gamma_1 = (a_1 a_{22} + a_2 a_{12})(a_{33}a_{44} - a_{34}a_{43})$ $\gamma_2 = a_3 a_{44} + a_4 a_{34}$ $\gamma_3 = (a_{11}a_{22} + a_{12}a_{21})(a_{33}a_{44} - a_{34}a_{43})$ $\gamma_4 = (a_1 a_{21} - a_2 a_{11})(a_{33}a_{44} - a_{34}a_{43})$

The present paper deals with the fully washed out state only. The stability of the other equilibrium states will be presented in the forth coming communications.

STABILITY OF THE FULLY WASHED OUT EQUILIBRIUM STATE

(Sl. No. 1 in the above Table)

To discuss the stability of equilibrium point

$$\overline{N_1} = 0, \overline{N_2} = 0, \overline{N_3} = 0, \overline{N_4} = 0$$

Let us consider small deviations u_1 (t), u_2 (t), u_3 (t), u_4 (t) from the steady state

i.e.,

$$N_i(t) = \overline{N_i} + u_i(t), i = 1, 2, 3, 4 \quad \text{........} \quad (6)$$

Where u_i (t) is a small perturbations in the species S_i.

Substituting (6) in (1), (2), (3), (4) and neglecting products and higher powers of u_1, u_2, u_3, u_4, we get

$$\frac{du_1}{dt} = a_1 u_1 \quad \text{......} \quad (7)$$

$$\frac{du_2}{dt} = a_2 u_2 \quad \text{........} \quad (8)$$

$$\frac{du_3}{dt} = a_3 u_3 \quad \text{......} \quad (9)$$

$$\frac{du_4}{dt} = a_4 u_4 \quad \text{........} \quad (10)$$

The characteristic equation of which is

$$(\lambda - a_1)(\lambda - a_2)(\lambda - a_3)(\lambda - a_4) = 0 \quad \text{........} \quad (11)$$

the roots a_1, a_2, a_3, a_4 of which are all positive.

Hence the Fully Washed out State is unstable.

The solutions of the equations (7), (8), (9), (10) are

$$u_1 = u_{10} e^{a_1 t} \quad \text{............} \quad (12)$$

$$u_2 = u_{20} e^{a_2 t} \quad \text{............} \quad (13)$$

$$u_3 = u_{30} e^{a_3 t} \quad \text{..........} \quad (14)$$

$$u_4 = u_{40} e^{a_4 t} \quad \text{.............} \quad (15)$$

Where u_{10}, u_{20}, u_{30}, u_{40} are the initial values of u_1, u_2, u_3, u_4 respectively.

There would arise in all 576 cases depending upon the ordering of the magnitudes of the growth rates a_1, a_2, a_3, a_4 and the initial values of the perturbations u_{10}(t), u_{20}(t), u_{30}(t),u_{40}(t) of the species S_1, S_2, S_3, S_4. Of these 576 situations some typical variations are illustrated through respective solution curves that would facilitate to make some reasonable observations.

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Case (i): If $u_{10}<u_{20}<u_{30}<u_{40}$, $a_1<a_2<a_3<a_4$

In this case prey (S_1) has the least natural birth rate and S_4 dominates the prey (S_1), predator (S_2) and the host (S_3) of S_1 in natural growth rate as well as in its population strength.

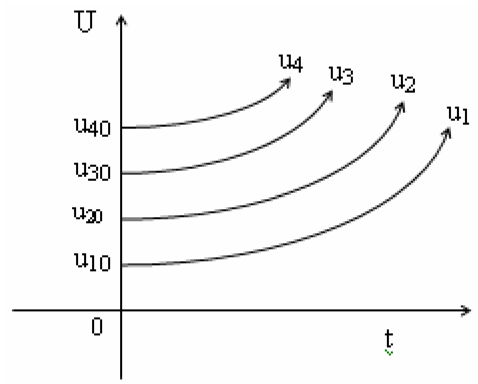

Figure-2

Case (ii): If $u_{10}<u_{20}<u_{30}<u_{40}$, $a_2<a_1<a_3<a_4$

In this case predator (S_2) has the least natural birth rate. Initially the predator (S_2) dominates over the prey (S_1) till the time instant $t_{21}^* = \frac{1}{a_1 - a_2}\log(\frac{u_{20}}{u_{10}})$ and there-after the prey (S_1) dominated the predator (S_2). The time t_{21}^* may be called the dominance time of the predator (S_2) over the prey (S_1).

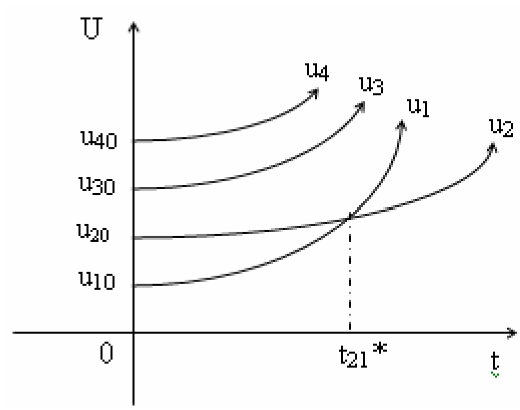

Figure-3

Case (iii): If $u_{10}<u_{30}<u_{40}<u_{20}$, $a_2<a_1<a_3<a_4$

In this case predator (S_2) has the least natural birth rate. Initially the predator (S_2) dominates over S_4, host (S_3) of S_1 and prey (S_1) till the time instant $t_{24}^*, t_{23}^*, t_{21}^*$ respectively and there after the dominance is reversed

Here

$$t_{24}^* = \frac{1}{a_2 - a_4}\log(\frac{u_{40}}{u_{20}}), t_{23}^* = \frac{1}{a_2 - a_3}\log(\frac{u_{30}}{u_{20}}), t_{21}^* = \frac{1}{a_1 - a_2}\log(\frac{u_{20}}{u_{10}})$$

Case (iv): If u20<u40<u30<u10, a3<a1<a2<a4

In this case the host (S_3) of S_1 has the least natural birth rate. Initially the host (S_3) of S_1 dominates over S_4 and the predator (S_2) till the times instant t^*_{34}, t^*_{32} respectively. Thereafter the dominance is reversed.

Here $t_{34}^* = \frac{1}{a_3 - a_4}\log(\frac{u_{40}}{u_{30}}); t_{32}^* = \frac{1}{a_2 - a_3}\log(\frac{u_{30}}{u_{20}})$

Also the prey (S_1) dominates over S_4, Predator (S_2) till the time instant t^*_{14}, t^*_{12} respectively and thereafter the dominance is reversed.

Here $t_{14}^* = \frac{1}{a_1 - a_4}\log(\frac{u_{40}}{u_{10}}); t_{12}^* = \frac{1}{a_1 - a_2}\log(\frac{u_{20}}{u_{10}})$

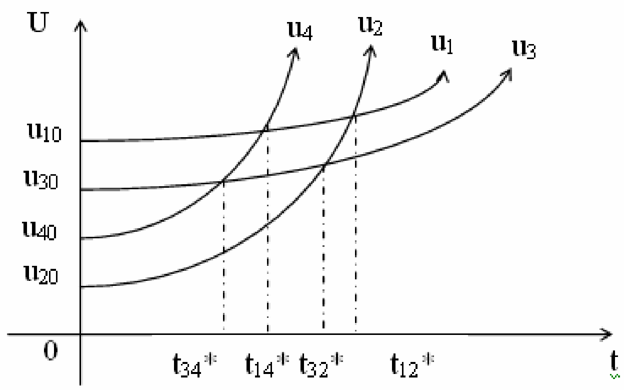

Figure-4

Case (v): If $u_{20}<u_{30}<u_{10}<u_{40}$, $a_3<a_2<a_4<a_1$

In this case the Host (S_3) of S_1 has the least natural birth rate. Initially it is dominates over the predator (S_2) till the time instant $t_{32}^* = \frac{1}{a_2 - a_3}\log(\frac{u_{30}}{u_{20}})$ and thereafter the dominance is reversed.

Also S_4 dominates over the prey (S_1) till the time instant $t_{41}^* = \frac{1}{a_1 - a_4}\log(\frac{u_{40}}{u_{10}})$ and thereafter the dominance is

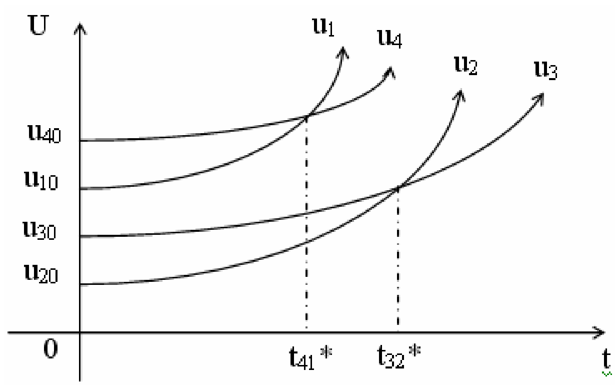

Figure-5

Case (vi): If $u_{20}<u_{30}<u_{40}<u_{10}$, $a_1<a_4<a_3<a_2$

In this case the prey (S_1) has the least natural birth rate. Initially the prey (S_1) dominates over its host, S_4 and Predator (S_2) till the time instant t^*_{13}, t^*_{14}, t^*_{12} respectively and thereafter the dominance is reversed.

Also S_4 dominates over the host (S_3) of S_1, and the predator (S_2) till the time instant t^*_{43}, t^*_{42} and thereafter the dominance is reversed.

Similarly, the host (S_3) of S_1 dominates over the predator (S_2) till the time instant t^*_{32} and the dominance gets reversed thereafter.

Here

$$t_{13}^* = \frac{1}{a_1 - a_3}\log(\frac{u_{30}}{u_{10}}); t_{14}^* = \frac{1}{a_1 - a_4}\log(\frac{u_{40}}{u_{10}}); t_{12}^* = \frac{1}{a_1 - a_2}\log(\frac{u_{20}}{u_{10}})$$

$$t_{43}^* = \frac{1}{a_3 - a_4}\log(\frac{u_{40}}{u_{30}}); t_{42}^* = \frac{1}{a_2 - a_4}\log(\frac{u_{40}}{u_{20}}); t_{32}^* = \frac{1}{a_2 - a_3}\log(\frac{u_{30}}{u_{20}})$$

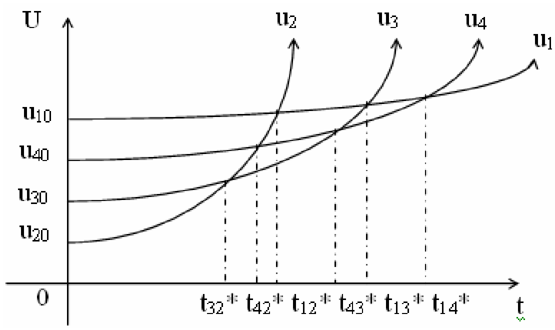

Figure-6

Case (vii): **If $u_{30}<u_{20}<u_{10}<u_{40}$, $a_3<a_4<a_1<a_2$**

In this case the host (S_3) of S_1 has the least natural birth rate. Initially S_4 dominates over both the prey (S_1) and predator (S_2) till the time instant t^*_{41}, t^*_{42} respectively and thereafter the dominance is reversed.

Also the Prey (S_1) dominates over the Predator (S_2) upto the time instant t^*_{12} and the dominance gets reversed after.

Here $t_{41}^* = \frac{1}{a_1 - a_4}\log(\frac{u_{40}}{u_{30}}); t_{42}^* = \frac{1}{a_2 - a_4}\log(\frac{u_{40}}{u_{20}})$

and $t_{12}^* = \frac{1}{a_1 - a_2}\log(\frac{u_{20}}{u_{10}})$

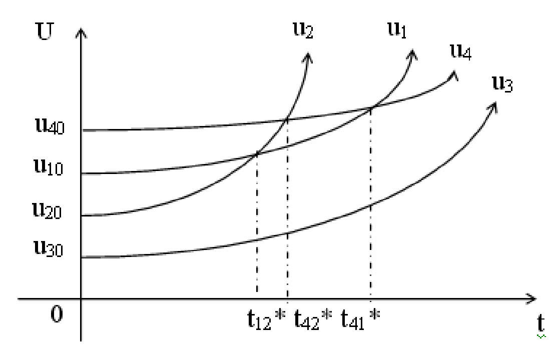

Figure-7

Case (viii): If $u_{30}<u_{10}<u_{20}<u_{40}$, $a_2<a_1<a_4<a_3$

In this case the predator (S_2) has the least natural birth rate. Initially the predator (S_2) dominates over the prey (S_1), host (S_3) of S_1 till the time instant t^*_{21}, t^*_{23} respectively and thereafter the dominance is reversed.

Also the prey (S_1) dominates over its host till the time instant t^*_{12} and thereafter the dominance is reversed. Similarly S_4 dominates over the host (S_3) of S_1 till the time instant t^*_{43} the dominance gets reversed after.

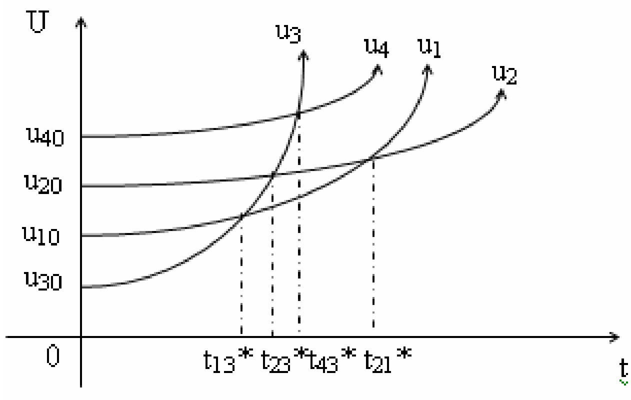

Figure-8

Here

$$t_{21}^* = \frac{1}{a_1 - a_2}\log(\frac{u_{20}}{u_{10}}); t_{23}^* = \frac{1}{a_2 - a_3}\log(\frac{u_{30}}{u_{20}})$$

$$t_{13}^* = \frac{1}{a_1 - a_3}\log(\frac{u_{30}}{u_{10}}); t_{43}^* = \frac{1}{a_3 - a_4}\log(\frac{u_{40}}{u_{30}})$$

Case (ix): If $u_{30}<u_{20}<u_{40}<u_{10}$, $a_1<a_4<a_3<a_2$

In this case the prey (S_1) has the least natural birth rate. Initially the prey (S_1) dominates over its host, predator (S_2) and S_4 till the time instant t^*_{13}, t^*_{12}, t^*_{14} respectively and there after the dominance is reversed. Also S_4 dominates over the predator (S_2) and the host (S_3) of S_1 till the times instant t^*_{42} and t^*_{43} and thereafter the dominance is reversed.

Here

$$t_{13}^* = \frac{1}{a_1 - a_3}\log(\frac{u_{30}}{u_{10}}); t_{12}^* = \frac{1}{a_1 - a_2}\log(\frac{u_{20}}{u_{10}}); t_{14}^* = \frac{1}{a_1 - a_4}\log(\frac{u_{40}}{u_{10}})$$

and $t_{42}^* = \frac{1}{a_2 - a_4}\log(\frac{u_{40}}{u_{20}}); t_{43}^* = \frac{1}{a_3 - a_4}\log(\frac{u_{40}}{u_{30}})$

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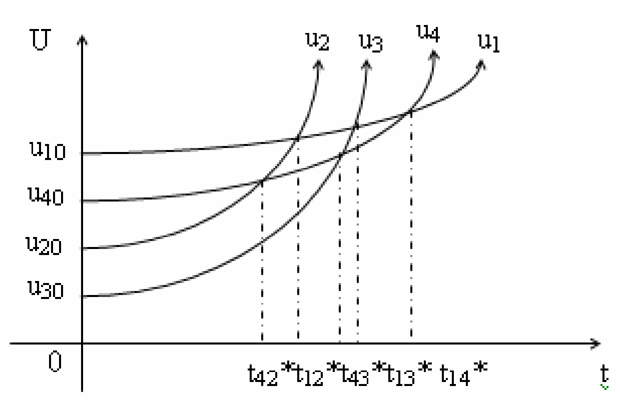

Figure-9

Case (x): If $u_{40}<u_{10}<u_{30}<u_{20}$, $a_1<a_2<a_4<a_3$

In this case the prey (S_1) has the least natural birth rate. Initially the prey (S_1) dominates over S_4 till the time instant t^*_{14} and thereafter the dominance is reversed.

Also the predator (S_2) dominates over S_3 and S_4 till the time instants t^*_{23}, t^*_{24} respectively and thereafter the dominance is reversed.

Here $t_{14}^* = \frac{1}{a_2 - a_3}\log(\frac{u_{40}}{u_{10}}); t_{24}^* = \frac{1}{a_2 - a_4}\log(\frac{u_{40}}{u_{20}})$

and $t_{23}^* = \frac{1}{a_2 - a_3}\log(\frac{u_{30}}{u_{20}})$

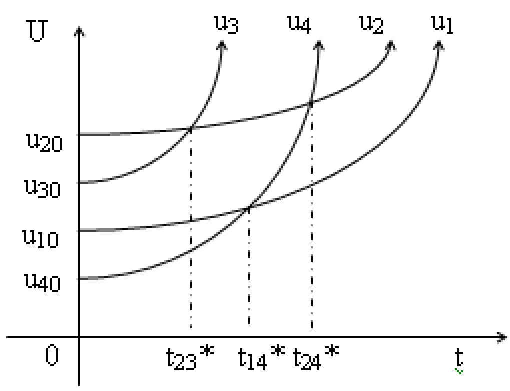

Figure-10

Case (xi): If $u_{40}<u_{10}<u_{20}<u_{30}$, $a_2<a_3<a_1<a_4$

In this case the predator (S_2) has the least natural birth rate. Initially the predator (S_2) dominates over S_4 and prey (S_1) till the time instant t^*_{24}, t^*_{21} respectively and thereafter the dominance is reversed.

Also the host (S_3) of S_1 dominates over the prey (S_1) and S_4 till the time instant t^*_{31}, t^*_{34} respectively and thereafter the dominance is reversed. Similarly the prey (S_1) dominates over S_4 till the time instant t^*_{14} and the dominance gets reversed after.

Here

$$t_{24}^* = \frac{1}{a_2 - a_4}\log(\frac{u_{40}}{u_{20}}); t_{21}^* = \frac{1}{a_1 - a_2}\log(\frac{u_{20}}{u_{10}})$$

$$t_{31}^* = \frac{1}{a_1 - a_3}\log(\frac{u_{30}}{u_{10}}); t_{34}^* = \frac{1}{a_3 - a_4}\log(\frac{u_{40}}{u_{30}})$$

and $t_{14}^* = \frac{1}{a_1 - a_4}\log(\frac{u_{40}}{u_{10}})$

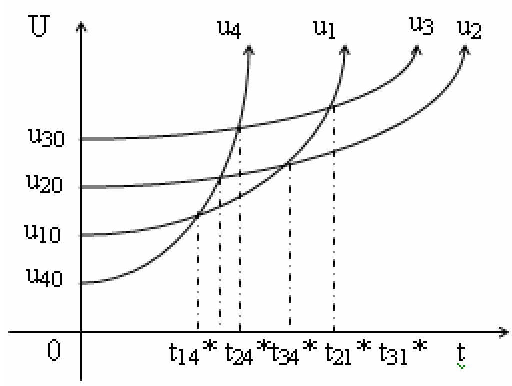

Figure-11

Case (xii): If $u_{40}<u_{20}<u_{30}<u_{10}$, $a_4<a_1<a_2<a_3$

In this case S_4 has the least natural birth rate. Initially the prey (S_1) dominates over its host and predator (S_2) till the time instant t^*_{13}, t^*_{12} respectively and thereafter the dominance is reversed.

Here $t_{13}^* = \frac{1}{a_1 - a_3}\log(\frac{u_{30}}{u_{10}}); t_{12}^* = \frac{1}{a_1 - a_2}\log(\frac{u_{20}}{u_{10}})$

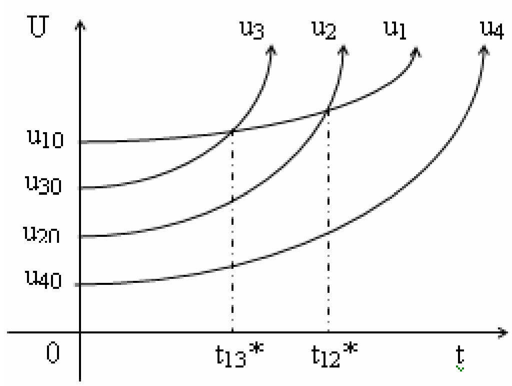

Figure-12

TRAJECTORIES OF PERTURBATIONS

The trajectories in the u_1-u_2 plane given by $(\frac{u_1}{u_{10}})^{a_2} = (\frac{u_2}{u_{20}})^{a_1}$ and are shown in Figure-13.

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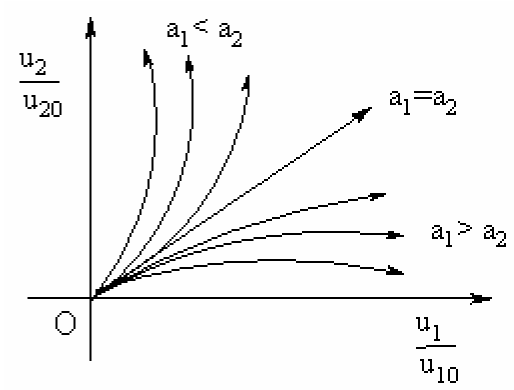

Figure-13

Also the trajectories in the $u_1 - u_3$ plane given by $(\frac{u_1}{u_{10}})^{a_3} = (\frac{u_3}{u_{30}})^{a_1}$ and are shown in Figure-14.

Similarly the trajectories in the u_1-u_4, u_2-u_3, u_2-u_4, u_3-u_4 planes are $(\frac{u_1}{u_{10}})^{a_4} = (\frac{u_4}{u_{40}})^{a_1}, (\frac{u_2}{u_{20}})^{a_3} = (\frac{u_3}{u_{30}})^{a_2}$, $(\frac{u_2}{u_{20}})^{a_4} = (\frac{u_4}{u_{40}})^{a_2}, (\frac{u_3}{u_{30}})^{a_4} = (\frac{u_4}{u_{40}})^{a_3}$ respectively.

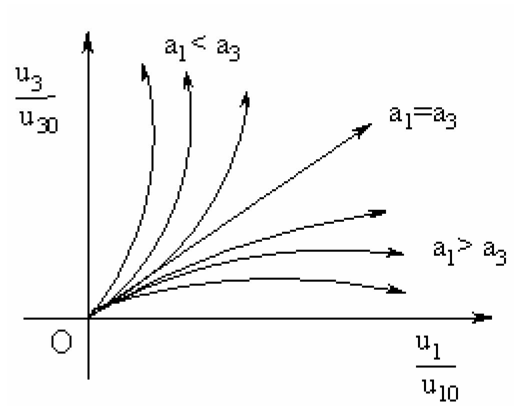

Figure-14

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