A MATHEMATICAL MODEL OF FOUR SPECIES SYN-ECOSYMBIOSIS COMPRISING OF PREY-PREDATION, MUTUALISM AND COMMENSALISMS-I (FULLY WASHED OUT STATE)

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ABSTRACT
This investigation deals with a mathematical model of a four species (S₁, S₂, S₃ and S₄) Syn-Ecological system (Fully Washed out State). S₂ is a predator surviving on the prey S₁; the prey is a commensal to the host S₃ which itself is in mutualism with the fourth species S₄; S₂ and S₄ are neutral. The mathematical model equations characterizing the syn-ecosystem constitute a set of four first order non-linear coupled differential equations. There are in all sixteen equilibrium points. Criteria for the asymptotic stability of one of the sixteen equilibrium points: the fully washed out state is established. The linearised equations for the perturbations over the equilibrium point are analyzed to establish the criteria for stability. The system is noticed to be locally stable. Trajectories of the perturbations have been illustrated.

Keywords: mathematical model, species, syn-ecological system, mutualism, commensalism, differential equations.

INTRODUCTION
Mathematical modeling is an important interdisciplinary activity which involves the study of some aspects of diverse disciplines. Biology, Epidemiology, Physiology, Ecology, Immunology, Bio-economics, Genetics, Pharmacokinetics are some of those disciplines. This mathematical modeling has raised to the zenith in recent years and spread to all branches of life and drew the attention of every one. Mathematical modeling of ecosystems was initiated by Lotka [8] and by Volterra [14]. The general concept of modeling has been presented in the treatises of Meyer [9], Cushing [2], Paul Colinvaux [10], Freedman [3], Kapur [5, 6]. The ecological interactions can be broadly classified as prey-predation, competition, mutualism and so on. N.C. Srinivas [13] studied the competitive eco-systems of two species and three species with regard to limited and unlimited resources. Later, Lakshmi Narayan [7] has investigated the two species prey-predator models. Recently stability analysis of competitive species was investigated by Archana Reddy [1]. Local stability analysis for a two-species ecological mutualism model has been investigated by B. Ravindra Reddy et al., [11, 12].

The present investigation is devoted to an analytical study of a four species Syn-Ecological system. S₁ is a predator surviving on the prey S₁; the prey is a commensal to the host S₃ which itself is in mutualism with the fourth species S₄; S₂ and S₄ are neutral. Figure-1 shows the Schematic Sketch of the system under investigation. The model equations of the system constitute a set of four first order non-linear ordinary differential coupled equations. In all the sixteen equilibrium points of the system are identified and the stability analysis is carried out only for the fully washed out state. The linearized perturbed equations over the equilibrium states are solved and the trajectories illustrated.

Figure-1. Schematic sketch of the Syn Eco-system.

BASIC EQUATIONS

Notation adopted

N₁(t): The Population of the Prey (S₁)
N₂(t): The Population of the Predator (S₂)
N₃(t): The Population of the Host (S₃) of the Prey (S₁) and mutual to S₄
N₄(t): The Population of S₄ mutual to S₃

a₈, a₉, a₁₀, a₁₁: Natural growth rates of S₁, S₂, S₃, S₄
a₁₂, a₁₃, a₁₄, a₁₅: Self inhibition coefficients of S₁, S₂, S₃, S₄
a₁₆: Interaction (Prey-Predator) coefficients of S₁ due to S₂, S₂ due to S₁
a₁₇: Coefficient for commensal for S₁ due to the Host S₃
a₁₈, a₁₉: Mutually interaction between S₃ and S₄

Further the variables N₁, N₂, N₃, N₄ are non-negative and the model parameters a₈, a₉, a₁₀, a₁₁, a₁₂, a₁₃, a₁₄, a₁₅, a₁₆, a₁₇, a₁₈, a₁₉ are assumed to be non-negative constants.
The model equations for the growth rates of $S_1$, $S_2$, $S_3$, $S_4$ are

\[
\frac{dN_1}{dt} = a_1 N_1 - a_{11} N_1^2 - a_{12} N_1 N_2 + a_{13} N_1 N_3 \quad \ldots \quad (1)
\]

\[
\frac{dN_2}{dt} = a_2 N_2 - a_{22} N_2^2 + a_{21} N_2 N_1 \quad \ldots \quad (2)
\]

\[
\frac{dN_3}{dt} = a_3 N_3 - a_{33} N_3^2 + a_{34} N_3 N_4 \quad \ldots \quad (3)
\]

\[
\frac{dN_4}{dt} = a_4 N_4 - a_{44} N_4^2 + a_{43} N_4 N_3 \quad \ldots \quad (4)
\]

**EQUILIBRIUM STATES**

The system under investigation has sixteen equilibrium states defined by

\[
\frac{dN_i}{dt} = 0, \quad i = 1, 2, 3, 4 \quad \ldots \quad (5)
\]

are given in the following table.

<table>
<thead>
<tr>
<th>S. No.</th>
<th>Equilibrium States</th>
<th>Equilibrium point</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Fully Washed out state</td>
<td>$N_1 = 0, N_2 = 0, N_3 = 0, N_4 = 0$</td>
</tr>
<tr>
<td>2</td>
<td>Only $S_4$ survives</td>
<td>$N_1 = 0, N_2 = 0, N_3 = 0, N_4 = \frac{a_4}{a_{44}}$</td>
</tr>
<tr>
<td>3</td>
<td>Only the host ($S_3$) of $S_1$ survives</td>
<td>$N_1 = 0, N_2 = 0, N_3 = 0, N_4 = 0$</td>
</tr>
<tr>
<td>4</td>
<td>Only the predator $S_2$ survives</td>
<td>$N_1 = 0, N_2 = \frac{a_2}{a_{22}}, N_3 = 0, N_4 = 0$</td>
</tr>
<tr>
<td>5</td>
<td>Only the prey $S_1$ survives</td>
<td>$N_1 = \frac{a_1}{a_{11}}, N_2 = 0, N_3 = 0, N_4 = 0$</td>
</tr>
<tr>
<td>6</td>
<td>Prey ($S_1$) and predator ($S_2$) washed out</td>
<td>$N_1 = 0, N_2 = 0, N_3 = \frac{a_1 a_{34} + a_2 a_{44}}{a_{33} a_{44} - a_{34} a_{43}}, N_4 = \frac{a_4 a_{43} + a_4 a_{33}}{a_{33} a_{44} - a_{34} a_{43}}$</td>
</tr>
<tr>
<td>7</td>
<td>Prey ($S_1$) and host ($S_3$) of $S_1$ washed out</td>
<td>$N_1 = 0, N_2 = \frac{a_2}{a_{22}}, N_3 = 0, N_4 = \frac{a_4}{a_{44}}$</td>
</tr>
<tr>
<td>8</td>
<td>Prey ($S_1$) and $S_4$ washed out</td>
<td>$N_1 = 0, N_2 = \frac{a_2}{a_{22}}, N_3 = \frac{a_3}{a_{33}}, N_4 = 0$</td>
</tr>
<tr>
<td>9</td>
<td>Predator ($S_2$) and Host ($S_3$) of $S_1$ washed out</td>
<td>$N_1 = \frac{a_1}{a_{11}}, N_2 = 0, N_3 = 0, N_4 = \frac{a_4}{a_{44}}$</td>
</tr>
<tr>
<td>10</td>
<td>Predator ($S_2$) and $S_4$ washed out</td>
<td>$N_1 = \frac{a_1 a_{33} + a_2 a_{13}}{a_{11} a_{33}}, N_2 = 0, N_3 = \frac{a_1}{a_{33}}, N_4 = 0$</td>
</tr>
<tr>
<td>11</td>
<td>Prey ($S_1$) and predator ($S_2$) survives</td>
<td>$N_1 = \frac{a_1 a_{22} - a_2 a_{12}}{a_{11} a_{22} + a_{12} a_{21}}, N_2 = \frac{a_1 a_{21} + a_2 a_{11}}{a_{11} a_{22} + a_{12} a_{21}}, N_3 = 0, N_4 = 0$</td>
</tr>
<tr>
<td>12</td>
<td>Only the prey ($S_1$) washed out</td>
<td>$N_1 = 0, N_2 = \frac{a_2}{a_{22}}, N_3 = \frac{a_1 a_{34} + a_2 a_{44}}{a_{33} a_{44} - a_{34} a_{43}}, N_4 = \frac{a_4 a_{43} + a_4 a_{33}}{a_{33} a_{44} - a_{34} a_{43}}$</td>
</tr>
<tr>
<td>13</td>
<td>Only the predator ($S_2$) washed out</td>
<td>$N_1 = \frac{a_1}{\alpha_1}, N_2 = 0, N_3 = \frac{a_4 a_{34} + a_4 a_{44}}{a_{33} a_{44} - a_{34} a_{43}}, N_4 = \frac{a_4 a_{34} + a_4 a_{33}}{a_{33} a_{44} - a_{34} a_{43}}$</td>
</tr>
</tbody>
</table>

where

$\alpha_1 = a_{13} (a_4 a_{34} + a_4 a_{44}) + a_4 (a_{33} a_{44} - a_{34} a_{43})$

$\alpha_2 = a_{11} (a_{33} a_{44} - a_{34} a_{43})$
The present paper deals with the fully washed out state only. The stability of the other equilibrium states will be presented in the forth coming communications.

**STABILITY OF THE FULLY WASHED OUT EQUILIBRIUM STATE**

(Sl. No. 1 in the above Table)

To discuss the stability of equilibrium point

\[ \overline{N}_1 = 0, \overline{N}_2 = 0, \overline{N}_3 = 0, \overline{N}_4 = 0 \]

Let us consider small deviations \( u_1(t), u_2(t), u_3(t), u_4(t) \) from the steady state i.e.,

\[ N_i(t) = \overline{N}_i + u_i(t), i = 1, 2, 3, 4 \]  

(6)

Where \( u_i(t) \) is a small perturbation in the species \( S_i \).

Substituting (6) in (1), (2), (3), (4) and neglecting products and higher powers of \( u_1, u_2, u_3, u_4 \), we get

\[
\frac{du_1}{dt} = a_1u_1 \quad \ldots \ldots \quad (7)
\]

\[
\frac{du_2}{dt} = a_2u_2 \quad \ldots \ldots \quad (8)
\]

\[
\frac{du_3}{dt} = a_3u_3 \quad \ldots \ldots \quad (9)
\]

\[
\frac{du_4}{dt} = a_4u_4 \quad \ldots \ldots \quad (10)
\]

The characteristic equation of which is

\[ (\lambda - a_1)(\lambda - a_2)(\lambda - a_3)(\lambda - a_4) = 0 \]  

(11)

the roots \( a_1, a_2, a_3, a_4 \) of which are all positive. Hence the Fully Washed out State is unstable.

The solutions of the equations (7), (8), (9), (10) are

\[ u_1(t) = u_{10}e^{a_1t} \quad \ldots \ldots \quad (12) \]

\[ u_2(t) = u_{20}e^{a_2t} \quad \ldots \ldots \quad (13) \]

\[ u_3(t) = u_{30}e^{a_3t} \quad \ldots \ldots \quad (14) \]

\[ u_4(t) = u_{40}e^{a_4t} \quad \ldots \ldots \quad (15) \]

Where \( u_{10}, u_{20}, u_{30}, u_{40} \) are the initial values of \( u_1, u_2, u_3, u_4 \) respectively.

There would arise in all 576 cases depending upon the ordering of the magnitudes of the growth rates \( a_1, a_2, a_3, a_4 \) and the initial values of the perturbations \( u_{10}(t), u_{20}(t), u_{30}(t), u_{40}(t) \) of the species \( S_1, S_2, S_3, S_4 \). Of these 576 situations some typical variations are illustrated through respective solution curves that would facilitate to make some reasonable observations.
Case (i): If \( u_{10} < u_{30} < u_{40} < u_{20}, \ a_1 < a_2 < a_3 < a_4 \)
In this case prey (S_1) has the least natural birth rate and S_4 dominates the prey (S_1), predator (S_2) and the host (S_3) of S_1 in natural growth rate as well as in its population strength.

![Figure-2](image)

Figure-2

Case (ii): If \( u_{10} < u_{20} < u_{30} < u_{40}, \ a_2 < a_1 < a_3 < a_4 \)
In this case predator (S_2) has the least natural birth rate. Initially the predator (S_2) dominates over the prey (S_1) till the time instant \( t_{21}^* = \frac{1}{a_1 - a_2} \log(\frac{u_{10}}{u_{20}}) \) and there-after the prey (S_1) dominated the predator (S_2). The time \( t_{21}^* \) may be called the dominance time of the predator (S_2) over the prey (S_1).

![Figure-3](image)

Figure-3

Case (iii): If \( u_{10} < u_{30} < u_{40} < u_{20}, \ a_2 < a_1 < a_3 < a_4 \)
In this case predator (S_2) has the least natural birth rate. Initially the predator (S_2) dominates over S_4, host (S_3) of S_1 and prey (S_1) till the time instant \( t_{24}^*, t_{23}^*, t_{21}^* \) respectively and there after the dominance is reversed. Here
\[
t_{21}^* = \frac{1}{a_1 - a_2} \log(\frac{u_{10}}{u_{20}}) = \frac{1}{a_2 - a_3} \log(\frac{u_{30}}{u_{40}}) = \frac{1}{a_3 - a_4} \log(\frac{u_{40}}{u_{30}})
\]

Also S_4 dominates over the prey (S_1) till the time instant \( t_{14}^* = \frac{1}{a_1 - a_4} \log(\frac{u_{10}}{u_{40}}) \) and thereafter the dominance is reversed.

![Figure-4](image)

Figure-4

Case (iv): If \( u_{20} < u_{40} < u_{30} < u_{10}, \ a_3 < a_1 < a_2 < a_4 \)
In this case the host (S_3) of S_1 has the least natural birth rate. Initially the host (S_3) dominates over S_4 and the predator (S_2) till the times instant \( t_{34}^*, t_{32}^* \) respectively. Thereafter the dominance is reversed.

Here \( t_{34}^* = \frac{1}{a_3 - a_4} \log(\frac{u_{30}}{u_{20}}) \); \( t_{32}^* = \frac{1}{a_2 - a_3} \log(\frac{u_{40}}{u_{20}}) \)

![Figure-5](image)

Figure-5

Case (v): If \( u_{20} < u_{30} < u_{10} < u_{40}, \ a_1 < a_2 < a_4 < a_3 \)
In this case the Host (S_3) of S_1 has the least natural birth rate. Initially it is dominates over the predator (S_2) till the time instant \( t_{32}^* = \frac{1}{a_2 - a_3} \log(\frac{u_{30}}{u_{20}}) \) and thereby the dominance is reversed. Also S_4 dominates over the prey (S_1) till the time instant \( t_{14}^* = \frac{1}{a_1 - a_4} \log(\frac{u_{10}}{u_{20}}) \) and thereafter the dominance is reversed.
Case (vi): If $u_{20} < u_{30} < u_{40} < u_{10}$, $a_1 < a_4 < a_3 < a_2$

In this case the prey ($S_1$) has the least natural birth rate. Initially the prey ($S_1$) dominates over its host, $S_4$ and predator ($S_2$) till the time instant $t^*_{13}$, $t^*_{14}$, $t^*_{12}$ respectively and thereafter the dominance is reversed.

Also $S_4$ dominates over the host ($S_1$) of $S_1$ and the predator ($S_2$) till the time instant $t^*_{43}$, $t^*_{42}$ and thereafter the dominance is reversed.

Similarly, the host ($S_3$) of $S_1$ dominates over the predator ($S_2$) till the time instant $t^*_{32}$ and the dominance gets reversed thereafter.

Here

\[ t^*_{13} = \frac{1}{a_1 - a_3} \log \left( \frac{u_{40}}{u_{10}} \right); t^*_{14} = \frac{1}{a_1 - a_4} \log \left( \frac{u_{30}}{u_{10}} \right); t^*_{12} = \frac{1}{a_1 - a_2} \log \left( \frac{u_{20}}{u_{10}} \right) \]

\[ t^*_{43} = \frac{1}{a_4 - a_3} \log \left( \frac{u_{30}}{u_{40}} \right); t^*_{42} = \frac{1}{a_4 - a_2} \log \left( \frac{u_{20}}{u_{40}} \right); t^*_{32} = \frac{1}{a_3 - a_2} \log \left( \frac{u_{20}}{u_{30}} \right) \]

Case (vii): If $u_{30} < u_{20} < u_{10} < u_{40}$, $a_3 < a_4 < a_2 < a_1$

In this case the host ($S_3$) of $S_1$ has the least natural birth rate. Initially $S_3$ dominates over both the prey ($S_1$) and predator ($S_2$) till the time instant $t^*_{41}$, $t^*_{42}$ respectively and thereafter the dominance is reversed.

Also the prey ($S_1$) dominates over the predator ($S_2$) upto the time instant $t^*_{12}$ and the dominance gets reversed after.

Here

\[ t^*_{41} = \frac{1}{a_1 - a_4} \log \left( \frac{u_{20}}{u_{40}} \right); t^*_{42} = \frac{1}{a_2 - a_4} \log \left( \frac{u_{30}}{u_{20}} \right) \]

and

\[ t^*_{12} = \frac{1}{a_1 - a_2} \log \left( \frac{u_{20}}{u_{10}} \right) \]

Case (viii): If $u_{30} < u_{10} < u_{20} < u_{40}$, $a_3 < a_1 < a_4 < a_2$

In this case the predator ($S_2$) has the least natural birth rate. Initially the predator ($S_2$) dominates over the prey ($S_1$), host ($S_3$) of $S_1$ till the time instant $t^*_{21}$, $t^*_{23}$ respectively and thereafter the dominance is reversed.

Also the prey ($S_1$) dominates over its host till the time instant $t^*_{12}$ and thereafter the dominance is reversed. Similarly $S_4$ dominates over the host ($S_3$) of $S_1$ till the time instant $t^*_{43}$ the dominance gets reversed after.

Here

\[ t^*_{21} = \frac{1}{a_1 - a_2} \log \left( \frac{u_{20}}{u_{10}} \right); t^*_{23} = \frac{1}{a_2 - a_3} \log \left( \frac{u_{30}}{u_{20}} \right) \]

and

\[ t^*_{43} = \frac{1}{a_3 - a_4} \log \left( \frac{u_{40}}{u_{30}} \right); t^*_{42} = \frac{1}{a_3 - a_2} \log \left( \frac{u_{20}}{u_{40}} \right) \]

Case (ix): If $u_{30} < u_{20} < u_{10} < u_{40}$, $a_1 < a_2 < a_3 < a_4$

In this case the prey ($S_1$) has the least natural birth rate. Initially the prey ($S_1$) dominates over its host, predator ($S_2$) and $S_4$ till the time instant $t^*_{13}$, $t^*_{12}$, $t^*_{14}$ respectively and thereafter the dominance is reversed.

Also $S_4$ dominates over the predator ($S_2$) and the host ($S_3$) of $S_1$ till the times instant $t^*_{42}$ and $t^*_{43}$ and thereafter the dominance is reversed.

Here

\[ t^*_{13} = \frac{1}{a_1 - a_3} \log \left( \frac{u_{20}}{u_{40}} \right); t^*_{12} = \frac{1}{a_2 - a_3} \log \left( \frac{u_{30}}{u_{10}} \right); t^*_{14} = \frac{1}{a_1 - a_4} \log \left( \frac{u_{30}}{u_{20}} \right) \]

and

\[ t^*_{42} = \frac{1}{a_2 - a_4} \log \left( \frac{u_{20}}{u_{40}} \right); t^*_{43} = \frac{1}{a_3 - a_4} \log \left( \frac{u_{40}}{u_{30}} \right) \]

Figure-6

Figure-7

Figure-8
Case (x): If $u_{40} < u_{10} < u_{30} < u_{20}$, $a_1 < a_2 < a_4 < a_3$

In this case the prey ($S_1$) has the least natural birth rate. Initially the prey ($S_1$) dominates over $S_4$ till the time instant $t_{14}^*$ and thereafter the dominance is reversed.

Also the predator ($S_2$) dominates over $S_3$ and $S_4$ till the time instants $t_{23}^*$, $t_{24}^*$ respectively and thereafter the dominance is reversed.

Here

$$t_{14}^* = \frac{1}{a_2-a_3} \log\left(\frac{u_{40}}{u_{10}}\right); t_{24}^* = \frac{1}{a_2-a_4} \log\left(\frac{u_{40}}{u_{20}}\right)$$

and

$$t_{23}^* = \frac{1}{a_2-a_3} \log\left(\frac{u_{40}}{u_{20}}\right)$$

Figure-9

Case (xi): If $u_{40} < u_{10} < u_{20} < u_{30}$, $a_4 < a_1 < a_2 < a_3$

In this case the predator ($S_2$) has the least natural birth rate. Initially the predator ($S_2$) dominates over $S_4$ and prey ($S_1$) till the time instant $t_{13}^*$ and thereafter the dominance is reversed.

Also the host ($S_3$) of $S_1$ dominates over the prey ($S_1$) and $S_4$ till the time instant $t_{34}^*$, $t_{31}^*$ respectively and thereafter the dominance is reversed. Similarly the prey ($S_1$) dominates over $S_4$ till the time instant $t_{14}^*$ and the dominance gets reversed after.

Here

$$t_{13}^* = \frac{1}{a_1-a_3} \log\left(\frac{u_{40}}{u_{10}}\right); t_{12}^* = \frac{1}{a_1-a_2} \log\left(\frac{u_{40}}{u_{20}}\right)$$

Figure-10

Case (xii): If $u_{40} < u_{20} < u_{30} < u_{10}$, $a_4 < a_1 < a_2 < a_3$

In this case $S_4$ has the least natural birth rate. Initially the prey ($S_1$) dominates over its host and predator ($S_2$) till the time instant $t_{13}^*$, $t_{12}^*$ respectively and thereafter the dominance is reversed.

Here

$$t_{13}^* = \frac{1}{a_1-a_3} \log\left(\frac{u_{40}}{u_{10}}\right); t_{12}^* = \frac{1}{a_1-a_2} \log\left(\frac{u_{40}}{u_{20}}\right)$$

Figure-11

TRAJECTORIES OF PERTURBATIONS

The trajectories in the $u_1$-$u_2$ plane given by

$$\left(\frac{u_1}{u_{10}}\right)^{a_1} = \left(\frac{u_2}{u_{20}}\right)^{a_2}$$

and are shown in Figure-12.
Also the trajectories in the $u_1-u_3$ plane given by

$\left(\frac{U_{10}}{u_{10}}\right)^{a_1} = \left(\frac{U_{30}}{u_{30}}\right)^{a_2}$ and are shown in Figure-14.

Similarly the trajectories in the $u_1-u_4$, $u_2-u_3$, $u_2-u_4$, $u_3-u_4$ planes are

$\left(\frac{U_{10}}{u_{10}}\right)^{a_1} = \left(\frac{u_{40}}{U_{40}}\right)^{a_3}$, $\left(\frac{U_{20}}{u_{20}}\right)^{a_2} = \left(\frac{U_{20}}{u_{20}}\right)^{a_3}$, $\left(\frac{U_{30}}{u_{30}}\right)^{a_3} = \left(\frac{U_{30}}{u_{30}}\right)^{a_4}$, $\left(\frac{U_{40}}{u_{40}}\right)^{a_4} = \left(\frac{U_{40}}{u_{40}}\right)^{a_4}$ respectively.

REFERENCES


