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SOME RESULT ON COMMON FIXED POINT THEOREM FOR A PAIR OF ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN GENERALIZED S-CONVEX METRIC SPACE

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ABSTRACT

Let X be a generalized s-convex metric space, and let Q , J be a pair of asymptotically nonexpansive mappings. In this paper, we will consider an Ishikawa type iteration process with errors to approximate the unique common fixed point of Q and J.

Keywords: s-convex functions, common fixed point, Nonexpansive mappings, convex metric space.

1. INTRODUCTION AND PRELIMINARIES

It is a well known fact that convexity and its generalization plays important role in different part of mathematics, mainly in optimization theory. In our paper we deal with a common generalization of s-convexity, approximate convexity, and results of Bernstein and Doetsch (1915). The concept of s-convexity and rational s-convexity was introduced by Breckner (1978).

Breckner (1978), H. Hudzik and L. Maligranda (1994) proved that s-convex functions are nonnegative, when 0 < s <1, moreover the set of s-convex functions increases as s decreases.

H. Hudzik and L. Maligranda (1994) discussed a few results connecting with s-convex functions in second sense and some new results about Hadamard's inequality for s-convex functions are discussed by M. Alomari, M. Darus (1977-1989), 2008 and U. S. Kirmaci (2007). S. S. Dragomir *et al.*, (1999) proved a variant of Hermite-Hadamard's inequality for s-convex functions in second sense.

Let (X,d) be a metric space, $Q,J : X \to X$ a pair of asymptotically nonexpansive mappings if there exists $a,b,c \in [0,1], a+2b+2c \le 1$ such that

$$d\left(Q^n x, J^n x\right) \le ad\left(x,y\right) + b\left[d\left(x,Q^n x\right)+d\left(y,J^n y\right)\right] + c\left[d\left(x,J^n y\right)+d\left(y,Q^n x\right)\right]$$

$(Wang, Li, Zhu, 2010)$

for all $x,y \in X, n \ge 1$.

Bose (1978) first defined a pair of mean nonexpansive mappings in Banach space, that is,

$$\|Qx-Jy\| \le a\|x-y\| + b\left[\|x-Qx\|+\|y-Jy\|\right] + c\left[\|x-Jy\|+\|y-Qx\|\right], \quad (1.1)$$

They proved several convergence theorems for common fixed points of mean nonexpansive mappings. Gu and Li (2008) also studied the same problem; they considered the Ishikawa iteration process to approximate the common fixed point of mean nonexpansive mappings in uniformly convex Banach space. Takahashi (1970) first introduced a notion of convex metric space, which is more general space, and each linear normed space is a special example of the space. Late on, Ciric (2003) proved the convergence of an Ishikawa type iteration process to approximate the common fixed point of a pair of mappings under condition B. Very recently, Wang and Liu (2009) give some sufficiency and necessary conditions for an Ishikawa type iteration process with errors to approximate a common fixed point of two mappings in generalized convex metric space.

Inspired and motivated by the above facts, we will consider the Ishikawa type iteration process with errors, which converges to the unique common fixed point of the pair of asymptotically nonexpansive mappings in generalized convex metric space. Our results extend and improve the corresponding results in (1970-2009).

First of all, we will need the following definitions and conclusions.

Definition 1.1 (see Takahashi, 1970).

Let (X,d) be a metric space, and $I=[0,1]$. A mapping $\omega : X^2 \times I \to X$ is said to be convex structure on X, if for any $(x,y,\alpha) \in X$ and $u \in X$, the following inequality holds:

$$d(\omega(x,y,\alpha),u) \le \alpha^s d(x,u) + (1-\alpha)^s d(y,u) \quad (1.2)$$

If (X,d) is a metric space with a convex structure ω, then (X,d) is called a convex metric space. Moreover, a nonempty subset E of X is said to be convex if $(x,y,\alpha) \in X$ for all $(x,y,\alpha) \in E^2 \times I$.

Definition 1.2 (see Y.-X. Tian, 2005).

Let (X,d) be a metric space, and $I=[0,1]$ and $\{a_n\},\{b_n\},\{c_n\}$ real sequences in [0,1] with $a_n+b_n+c_n=1$. A mapping $\omega: X^3\times I^3\to X$ is said to be convex structure on X, if for any $(x,y,z,a_n,b_n,c_n)\in X^3\times I^3$ and $u\in X$, the following inequality holds:

$$d\left(\omega(x,y,z,a_n,b_n,c_n),u\right)\le (a_n)^s d(x,u)+(b_n)^s d(y,u) + (c_n)^s d(z,u) \quad (1.3)$$

If (X,d) is a metric space with a convex structure ω, then (X,d) is called a generalized convex metric space. Moreover, a nonempty subset E of X is said to be convex if $\omega(x,y,z,a_n,b_n,c_n)\in E$, for all $(x,y,z,a_n,b_n,c_n)\in E^3\times I^3$.

Remark 1.3.

It is easy to see that every generalized convex metric space is a convex metric space $(let\ c_n=0)$.

Definition 1.4.

Let (X,d) be a generalized convex metric space with a convex structure $\omega: X^3\times I^3\to X$ and E a nonempty closed convex subset of X. Let $Q,J:E\to E$ be a pair of asymptotically nonexpansive mappings, and $\{a_n\},\{b_n\},\{c_n\}$ $\{a'_n\},\{b'_n\},\{c'_n\}$ six sequences in $[0,1]$ with $a_n+b_n+c_n = (a'_n)^p+(b'_n)^p+(c'_n)^p=1$, for any given $x_1\in E$, $n=1,2.........$ define a sequence $\{x_n\}$ as follows:

$$x_{n+1}=\omega(x_n,Q^n y_n,u_n,a_n,b_n,c_n)$$

$$y_n=\omega(x_n,J^n x_n,v_n,a'_n,b'_n,c'_n) \quad (1.4)$$

where $\{u_n\},\{v_n\}$ are two sequences in E satisfying the following condition. If for any nonnegative integers $n,m,1\le n<m,\delta(A_{nm})>0,$ then

$$\max_{n\le i,j\le m}\left\{\begin{matrix} d(x,y): x\in\{u_i,v_i\}, \\ y\in\{x_j,y_j,Qy_j,Jx_j,u_j,v_j\}\end{matrix}\right\}<\delta(A_{nm}), \quad (1.5)$$

$$(A_{nm})=\{x_i,y_i,Qy_i,Jx_i,u_i,v_i: n\le i\le m\},$$
$$\delta(A_{nm})=\sup_{x,y\in A_{nm}} d(x,y), \quad (1.6)$$

then $\{x_n\}$ is called the Ishikawa type iteration process with errors of a pair of asymptotically nonexpansive mappings Q and J.

Remark 1.5.

Note that the iteration processes considered in [1, 2, 4, 6] can be obtained from the above process as special cases by suitably choosing the space, the mappings, and the parameters.

Theorem 1.6. (see Wang and Liu 2009).

Let E be a nonempty closed convex subset of complete convex metric space X and $Q,J:E\to E$ uniformly quasi-Lipschitzian mappings with $L>0$ and $L'>0\ and\ F=F(Q)\cap F(J) \neq \left(F(J)=\{x\in X: Jx=x\}\right)$, Suppose that $\{x_n\}$ is the Ishikawa type iteration process with errors defined by [1.4], $\{u_n\}$, $\{v_n\}$ satisfy (1.5), and $\{a_n\},\{b_n\},\{c_n\}$ $\{a'_n\},\{b'_n\},\{c'_n\}$ are six sequences in $[0,1]$ satisfying

$$a_n+b_n+c_n=a'_n+b'_n+c'_n=1,\qquad \sum_{n=0}^{\infty}(b_n+c_n)<\infty, \quad (1.7)$$

then $\{x_n\}$ converge to a fixed point of S and T if and only if $\lim\ \inf_{n\to\infty} d(x_n,F)=0,$ where $d(x,F)=\inf\{d(x,p): p\in F\}$.

Remark 1.7.

Let $F(J)=\{x\in X: Jx=x\}\ne\emptyset.$ A mapping $J:X\to X$ is called uniformly quasi-Lipshitzian if there exists $L>0$ such that

$$d(J^n x,p)\le Ld(x,p) \quad (1.8)$$

for all $x\in X, p\in F(J), n\ge 1.$

2. MAIN RESULTS

Now, we will prove the strong convergence of the iteration scheme (1.4) to the unique common fixed point of a pair of asymptotically nonexpansive mappings S and T in complete generalized convex metric spaces.

Theorem 2.1.

Let E be a nonempty closed convex subset of complete generalized convex metric space X, and $Q,J:E\to E$ a pair of asymptotically nonexpansive mappings with $b\ne 0, and\ F=F(Q)\cap F(J)\ne\emptyset.$, Suppose $\{x_n\}$ as in (1.4), $\{u_n\}$, $\{v_n\}$ satisfy (1.5), and $\{a_n\},\{b_n\},\{c_n\}$ $\{a'_n\},\{b'_n\},\{c'_n\}$ are six sequences in $[0,1]$ satisfying

$$a_n + b_n + c_n = a'_n + b'_n + c'_n = 1, \quad \sum_{n=0}^{\infty}(b_n + c_n) < \infty, \tag{2.1}$$

then $\{x_n\}$ converge to the unique common fixed point of S and T if and only if $\lim\ \inf_{n\to\infty} d(x_n, F) = 0$, where $d(x,F) = \inf\{d(x,p) : p \in F\}$.

Proof.

The necessity of conditions is obvious. Thus, we will only prove the sufficiency. Let $p \in F$, *for all* $x \in E$,

$$d\left(Q^n x, p\right) \le a^s d\left(x,p\right) + b^s\left[d\left(x, Q^n x\right) + d\left(p,p\right)\right] + c^s\left[d\left(x,p\right) + d\left(p, Q^n x\right)\right]$$

$$\le a^s d\left(x,p\right) + b^s\left[d\left(x,p\right) + d\left(p,Q^n x\right)\right] + c^s\left[d\left(x,p\right) + d\left(p,Q^n x\right)\right] \tag{2.2}$$

implies

$$\left(1 - b^s - c^s\right) d\left(Q^n x, p\right) \le (a^s + b^s + c^s) d\left(x,p\right) \tag{2.3}$$

which yield (using the fact that $a + 2b + 2c \le 1 and\ b \ne 0$)

$$d\left(Q^n x, p\right) \le K d\left(x,p\right), \tag{2.4}$$

where $0 < K = (a+b+c)/(1-b-c) \le 1$.. Similarly, we also have $d\left(J^n x, p\right) \le K d\left(x,p\right)$,

By Remark 1.7, we get that *Q* and *J* are two uniformly quasi-Lipschitzian mappings$(with\ L = L' = K > 0)$. Therefore, from Theorem 1.6, we know that $\{x_n\}$ converges to a common fixed point of *Q* and *J*.

Finally, we prove the uniqueness. Let $p_1 = Sp_1 = Tp_1, p_2 = Sp_2 = Tp_2$, then by (Wang. Li, Zhu, 2010), we have

$$d\left(p_1,p_2\right) \le a^s d\left(p_1,p_2\right) + b^s\left[d\left(p_1,p_1\right) + d\left(p_2,p_2\right)\right] + c^s\left[d\left(p_1,p_2\right) + d\left(p_1,p_2\right)\right]$$

$$\le (a^s + 2c^s) d\left(p_1,p_2\right). \tag{2.5}$$

Since $(a+2c) < 1$, we obtain $p_1 = p_2$. This completes the proof.

Remark 2.2.

(i) We consider a sufficient and necessary condition for the Ishikawa type iteration process with errors in complete generalized convex metric space; our mappings are the more general mappings (a pair of asymptotically nonexpansive mappings), so our result extend and generalize the corresponding results in (1970,1978,2003,2008 and 2009).

(ii) Since $\{x_n\}$converges to the unique fixed point of S and T, we have improved
Theorem 1.6 in (Wang C. and L. W. Liu 2009).

Corollary 2.3.

Let E be a nonempty closed convex subset of Banach space X, $Q, J : X \to X$ a pair of asymptotically nonexpansive mappings, that is,

$$\left\|Q^n x - J^n x\right\| \le a\left\|x - y\right\| + b\left[\left\|x - Q^n x\right\| + \left\|y - J^n y\right\|\right] + c\left[\left\|x - J^n y\right\| + \left\|y - Q^n x\right\|\right] \tag{2.6}$$

with $b \ne 0, and\ F = F(Q) \cap F(J) \ne \emptyset.$ For any given $x_1 \in E, \{x_n\}$is an Ishikawa type iteration process with errors defined by

$$x_{n+1} = a_n x_n + b_n Q^n y_n + c_n u_n)$$
$$y_n = a'_n x_n + b'_n J^n y_n + c'_n v_n) \tag{2.7}$$

Where$\{u_n\}, \{v_n\} \in E$ are two bounded sequences and $\{a_n\}, \{b_n\}, \{c_n\}$ $\{a'_n\}, \{b'_n\}, \{c'_n\}$ are six sequences in $[0,1]$ satisfying

$$a_n + b_n + c_n = a'_n + b'_n + c'_n = 1, \quad \sum_{n=0}^{\infty}(b_n + c_n) < \infty, \tag{2.8}$$

Then, $\{x_n\}$converges to the unique common fixed point of *Q* and *J* if and only if $\lim\ \inf_{n\to\infty} d(x_n, F) = 0$, where $d(x,F) = \inf\{d(x,p) : p \in F\}$.

Proof. From the proof of Theorem 2.1, we have

$$\left\|Q^n x - p\right\| \le K\left\|x - p\right\|, \left\|J^n x - p\right\| \le K\left\|x - p\right\| \tag{2.9}$$

Where $K = (a+b+c)/(1-b-c)$. Hence, S and T are two uniformly quasi-Lipschitzian mappings in Banach space. Since Theorem 1.6 also holds in Banach spaces, we can prove that there exists a $p \in F$ such that $\lim_{n\to\infty}\left\|x_n - p\right\| = 0,$. The proof of uniqueness is the same to that of Theorem 2.1. Therefore,

$\{x_n\}$ converges to the unique common fixed point of Q and J.

Corollary 2.4.

Let E be a nonempty closed convex subset of Banach space X, $Q, J : X \to X$ a pair of asymptotically nonexpansive mappings, that is,

$$\|Q^n x - J^n x\| \le a\|x - y\| + b\left[\|x - Q^n x\| + \|y - J^n y\|\right] + c\left[\|x - J^n y\| + \|y - Q^n x\|\right] \quad (2.10)$$

with $b \neq 0, and\ F = F(Q) \cap F(J) \neq \emptyset.$ For any given $x_1 \in E, \{x_n\}$ is an Ishikawa type iteration process with errors defined by

$$x_{n+1} = \alpha_n^s x_n + (1 - \alpha_n^s) Q^n y_n),$$
$$y_n = \beta_n^s x_n + (1 - \beta_n^s) J^n x_n), \quad (2.11)$$

where $\{\alpha_n\}\{\beta_n\}$ are two sequences in [0,1] satisfying $\sum_{n=1}^{\infty}(1 - \alpha_n) < \infty$. Then, $\{x_n\}$ converges to the unique common fixed point of S and T if and only if $\lim\ \inf_{n\to\infty} d(x_n, F) = 0,$ where $d(x, F) = \inf\{\|x - p\| : p \in F\}.$

Proof.

Let $a_n = \alpha_n\ and\ a'_n = \beta_n\ and\ c_n = c'_n$. The result can be deduced immediately from Corollary 2.3. This completes the proof.

CONCLUSIONS

I. For $s = 1$, it becomes work of Convergence Theorems for the Unique Common Fixed Point of a Pair of Asymptotically Nonexpansive Mappings in Generalized Convex Metric Space.
II. We established Unique Common Fixed Point Theorem for a Pair of Asymptotically Nonexpansive Mappings in Generalized s-Convex Metric Space.

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