A HOST-COMMENSAL ECO-SYSTEM WITH HOST HARVESTING AT A CONSTANT RATE AND MORTALITY RATE FOR THE COMMENSAL

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ABSTRACT

This paper deals with a commensal-host ecological model with the host being harvested at a constant rate. Further, both the commensal and the host species are with limited resources. The Mathematical equations of the Model are characterized by a couple of first order non-linear ordinary differential equations. All the possible, six equilibrium points for the model are identified. Analytical solutions for the linearized perturbed equations are found and results are illustrated. Further, some threshold results are stated followed by the identification of threshold regions through illustrations. Criteria for global stability of linearized equations are discussed employing a property constructed Liapunov's function.

Keywords: equilibrium point, equilibrium state, stability, carrying capacity, reversal time of dominance.

1. INTRODUCTION

Ecosystem models are a development of theoretical ecology that aims to characterize the major dynamics of ecosystems, both to synthesis the understanding of such systems and to allow predictions of their behaviour (in general terms, or in response to particular changes). Research in theoretical ecology was initiated by Lotka [11] and by Volterra [18]. Since then many mathematicians and ecologists contributed to the growth of this area of knowledge as reported in the treatises of Mayer [12], Kapur [6, 7], Svirezhev and Logofet [17], Kushing [8] and Freedman [5]. The ecological interactions can be broadly classified as prey-predation, competition, commensalism, Ammensalism, Neutralism and so on. N.C. Srinivas [16] studied the competitive ecosystems of two species and three species with limited and unlimited resources. Later Lakshmi Narayan and Pattabhi Ramacharyulu [9, 10] investigated prey-predator ecological models with a partial cover for the prey and alternative food for the predator and prey-predator model with cover for the prey and alternate food for the predator and time delay. Stability analysis of competitive species was carried out by Archana Reddy, Pattabhi Ramacharyulu and Gandhi [1, 2], Bhaskara Rama Sarma and Pattabhi Ramacharyulu [3, 4]. While the mutualism between two species was examined by Ravindra Reddy [14]. Recently Phanikumar *et. al.*, [13] obtained the criteria for the stability of a Host- A flourishing Commensal species pair with limited resources. Seshagiri Rao *et. al.*, [15] investigated the stability of a host- A decaying commensal species pair with limited resources.

The present investigation is on a commensal-host ecological model with the host being harvested at a constant rate and both the species are with limited resources. The Mathematical equations of the Model are characterized by a couple of first order non-linear ordinary differential equations. In all, six equilibrium points for the model are identified. Analytical solutions for the linearized perturbed equations are found and results are illustrated. Further, some threshold results are stated followed by the identification of threshold regions through illustrations. Criteria for global stability of linearized equations are discussed employing a property constructed Liapunov's function.

2. BASIC EQUATIONS

Notation adopted

$N_1(t)$: Population of the Commensal species (S_1).

$N_2(t)$: Population of the Host species (S_2).

$e_1(= d_1 / a_{11})$: Extinction coefficient of S_1.

$c(= a_{12} / a_{11})$: Coefficient of Commensal.

$k_2(= a_2 / a_{22})$: Carrying capacity of S_2.

H_2 : Constant harvesting rate of S_2.

Further both the variables N_1 (t) and N_2 (t) are non-negative for all t and all the model parameters d_1, a_2, a_{11}, a_{22}, a_{12}, H_1 and H_2 are assumed to be non-negative constants.

Employing the above terminology, the equations for this model are given by the following system of non-linear coupled ordinary differential equations.

(i). Equation for the growth rate of the Commensal species (S_1) is:

$$\frac{dN_1}{dt} = a_{11}N_1\left[-e_1 - N_1 + cN_2\right] \tag{2.1}$$

(ii). Equation for the growth rate of the Host species (S_2) is:

$$\frac{dN_2}{dt} = a_{22}\left[k_2N_2 - N_2^{\,2} - H_2\right] \tag{2.2}$$

3. EQUILIBRIUM STATES

The system under investigation has the following six equilibrium states given by $\frac{dN_1}{dt} = 0$ and $\frac{dN_2}{dt} = 0$. These states are classified into two categories A and B.

A. The states in which only the host survives

A.1. When $k_2^{\,2} = 4H_2$ (A.1)

$$E_1: \overline{N_1} = 0;\ \overline{N_2} = \frac{k_2}{2} \tag{3.1}$$

A.2. When $k_2^{\,2} > 4H_2$ (A.2)

$$E_2: \overline{N_1} = 0;\ \overline{N_2} = \frac{k_2 + \sqrt{k_2^{\,2} - 4H_2}}{2} \tag{3.2}$$

$$E_3: \overline{N_1} = 0;\ \overline{N_2} = \frac{k_2 - \sqrt{k_2^{\,2} - 4H_2}}{2} \tag{3.3}$$

B. The co-existent states

B.1. When $k_2^{\,2} = 4H_2$ (B.1)

$$E_4: \overline{N_1} = \frac{ck_2 - 2e_1}{2};\ \overline{N_2} = \frac{k_2}{2} \tag{3.4}$$

This exists only when $ck_2 > 2e_1$.

B.2. When $k_2^{\,2} > 4H_2$ (B.2)

$$E_5: \overline{N_1} = \frac{c\left[k_2 + \sqrt{k_2^{\,2} - 4H_2}\right]}{2} - e_1;\quad \overline{N_2} = \frac{k_2 + \sqrt{k_2^{\,2} - 4H_2}}{2} \tag{3.5}$$

$$E_6: \overline{N_1} = \frac{c\left[k_2 - \sqrt{k_2^{\,2} - 4H_2}\right]}{2} - e_1;\quad \overline{N_2} = \frac{k_2 - \sqrt{k_2^{\,2} - 4H_2}}{2} \tag{3.6}$$

Both these exists only when $\frac{c\left[k_2 + \sqrt{k_2^{\,2} - 4H_2}\right]}{2} > e_1$

4. STABILITY OF THE EQUILIBRIUM STATES

Let N = (N_1, N_2) = $\overline{N} + U$ (4.1)

where $U = (u_1, u_2)$ is a perturbation over the equilibrium state $\overline{N} = (\overline{N_1}, \overline{N_2})$ are so small that their second and higher powers and products are neglected.

The basic equations (2.1) and (2.2) are linearized to obtain the equations for the perturbed state.

$$\frac{dU}{dt} = AU \tag{4.2}$$

where

$$A = \begin{bmatrix} -e_1 a_{11} - 2a_{11}\overline{N_1} + ca_{11}\overline{N_2} & ca_{11}\overline{N_1} \\ 0 & k_2 a_{22} - 2a_{22}\overline{N_2} \end{bmatrix} \tag{4.3}$$

The eigen values of the characteristic matrix A are:

$$(\lambda_1, \lambda_2) = \left(-e_1 a_{11} - 2a_{11}\overline{N_1} + ca_{11}\overline{N_2},\ k_2 a_{22} - 2a_{22}\overline{N_2}\right) \tag{4.4}$$

The equilibrium state is stable, only when both the eigen values of the characteristic matrix A are negative in case they are real or both the roots have negative real parts in case they are complex.

4.1. Stability of the equilibrium state $E_1: \overline{N_1} = 0;\ \overline{N_2} = \frac{k_2}{2}$

In this case the corresponding linearized perturbed equations are:

$$\frac{d}{dt}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} a_{11}\left(\frac{ck_2}{2} - e_1\right) & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{4.5}$$

From (4.5), the corresponding characteristic equation is:

$$\lambda\left[\lambda - a_{11}\left(\frac{ck_2}{2} - e_1\right)\right] = 0 \tag{4.6}$$

The characteristic roots of (4.6) are $\lambda_1 = a_{11}\left(\frac{ck_2}{2} - e_1\right)$ and $\lambda_2 = 0$. Since one of the two roots is zero, this state is unstable.

Here three cases will arise, these are:

CASE 1A: $\frac{ck_2}{2} > e_1$; CASE 1B: $\frac{ck_2}{2} < e_1$; CASE 1C: $\frac{ck_2}{2} = e_1$

Case-1A: When $\frac{ck_2}{2} > e_1$

From (4.5), the solutions of the linearized perturbed equations in this case are given by:

$$u_1 = u_{10} e^{a_{11}\left(\frac{ck_2}{2} - e_1\right)t} \tag{4.7}$$

$$u_2 = u_{20} \tag{4.8}$$

and these solution curves are illustrated as follows.

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Case -1A.1: When $u_{10} > u_{20}$

The initial population strength of the commensal is greater than that of the host i.e., $u_{10} > u_{20}$. In this case the commensal always out-numbers the host. Further the host species is observed to be at a constant distance from the equilibrium point in the course of time, while the commensal species goes far away from the equilibrium point is shown in Figure-1.

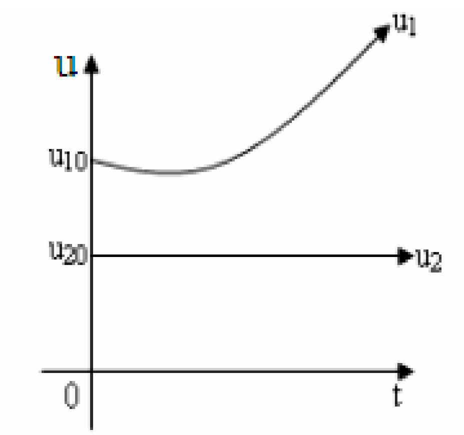

Figure-1

Case-1A.2: When $u_{10} < u_{20}$

The initial population strength of the host is greater than that of the commensal i.e., $u_{10} < u_{20}$. In this case $u_1(t) = u_2(t)$ is possible at a time $t^* = \frac{2}{a_{11}(ck_2 - 2e_1)} \log\left(\frac{u_{20}}{u_{10}}\right)$. This is the dominance reversal time as shown in Figure-2.

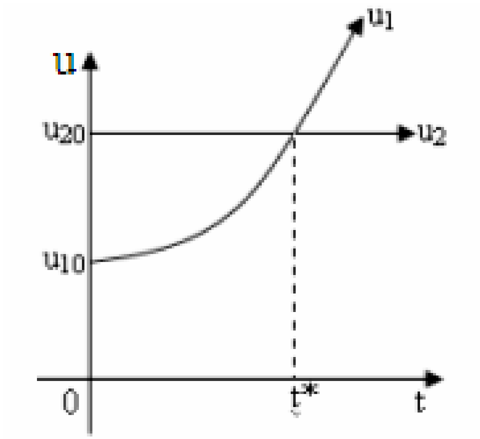

Figure-2

4.1 (a) Trajectories of the perturbed species

Eliminating 't' between the equations (4.7) and (4.8), we obtain:

$$\frac{u_2}{u_{20}} = 1 \quad (4.9)$$

and the trajectory is a straight line as shown in Figure-3.

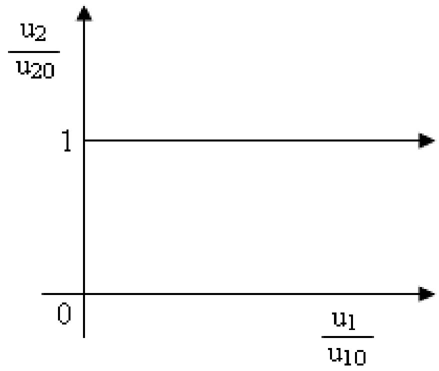

Figure-3

Case-1B: When $\frac{ck_2}{2} < e_1$

From (4.5), the solutions of the linearized perturbed equations in this case are given by:

$$u_1 = u_{10} e^{-a_{11}\left(e_1 - \frac{ck_2}{2}\right)t} \quad (4.10)$$

$$u_2 = u_{20} \quad (4.11)$$

and the solution curves of (4.10) and (4.11) are illustrated below.

Case-1B.1: When $u_{10} > u_{20}$

The initial population strength of the commensal is greater than that of the host i.e., $u_{10} > u_{20}$. In this case $u_1(t) = u_2(t)$ is possible at a time $t^* = \frac{2}{a_{11}(2e_1 - ck_2)} \log\left(\frac{u_{10}}{u_{20}}\right)$. This is the dominance reversal time over the host as shown in Figure-4.

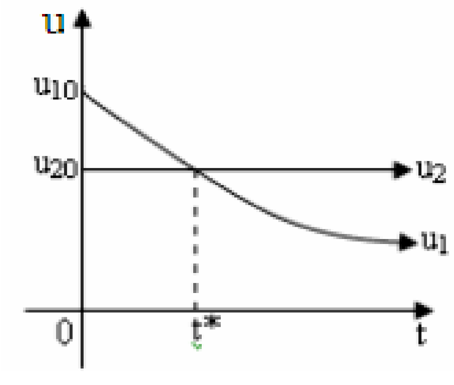

Figure-4

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Case-1B.2: When $u_{10} < u_{20}$

In this case the host continues out-numbering the commensal through out its natural growth rate as shown in Figure-5. However the commensal converge asymptotically to the equilibrium point, while the host is observed to be at a constant distance from the equilibrium point in the course of time.

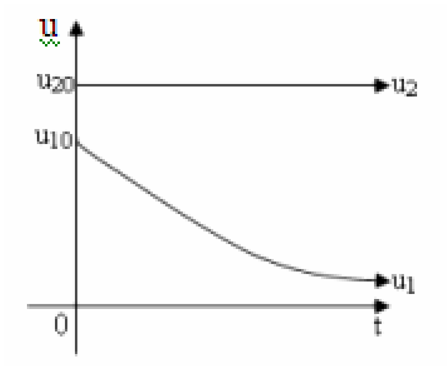


Figure-5

4.1 (b) Trajectories of the perturbed species

Eliminating 't' between the equations (4.10) and (4.11), we obtain:

$$\frac{u_2}{u_{20}} = 1 \qquad (4.12)$$

and the trajectory is a straight line as shown in Figure-6.

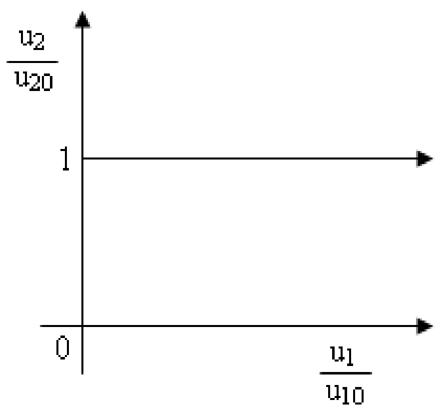


Figure-6

Case-1C: When $\frac{ck_2}{2} = e_1$

The solutions of the linearized perturbed equations in this case are given by:

$$u_1 = u_{10} \qquad (4.13)$$

$$u_2 = u_{20} \qquad (4.14)$$

and the solution curves of (4.13) and (4.14) are illustrated below.

Case-1C.1: When $u_{10} > u_{20}$

The initial population strength of the commensal is greater than that of the host. In this case the commensal always out-numbers the host. Further both the species are at a constant distance from the equilibrium point as shown in Figure-7.

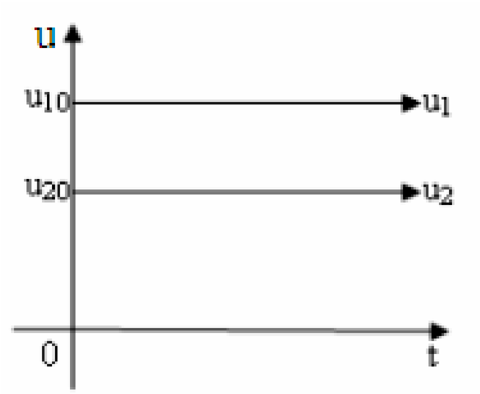


Figure-7

Case-1C.2: When $u_{10} < u_{20}$

The initial population strength of the host is greater than that of the commensal i.e., $u_{10} < u_{20}$. In this case the host always out-numbers the commensal as shown in Figure-8.

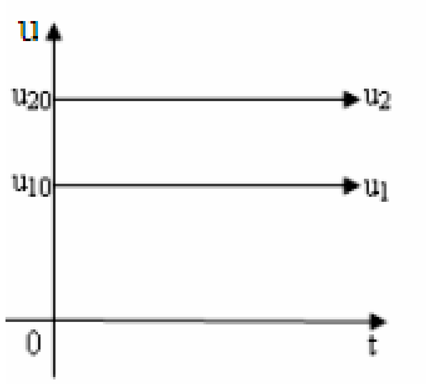


Figure-8

4.1 (c) Trajectories of the perturbed species

Eliminating't' between the equations (4.13) and (4.14), we obtain:

$$\frac{u_1}{u_{10}} = \frac{u_2}{u_{20}} \qquad (4.15)$$

and the trajectory is a straight line as shown in Figure-9.

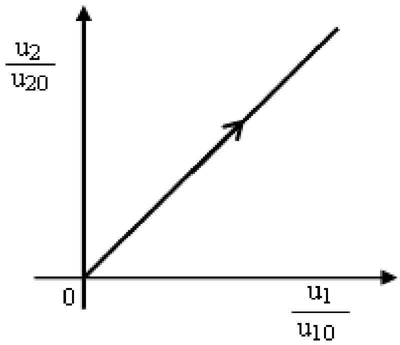


Figure-9

4.2. Stability of the equilibrium state

$E_2 : \overline{N_1} = 0 \; ; \; \overline{N_2} = \frac{k_2 + \sqrt{k_2^{\,2} - 4H_2}}{2}$

In this case the characteristic roots of the perturbed equations are

$$\lambda_1 = a_{11}\left[c\left(\frac{k_2 + \sqrt{k_2^{\,2} - 4H_2}}{2}\right) - e_1 \right]$$

and $\lambda_2 = -a_{22}\sqrt{k_2^{\,2} - 4H_2}$.

As before three cases will arise.

Case-2A: When $e_1 > c\left(\frac{k_2 + \sqrt{k_2^{\,2} - 4H_2}}{2}\right)$

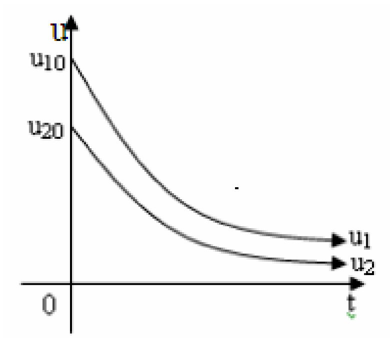
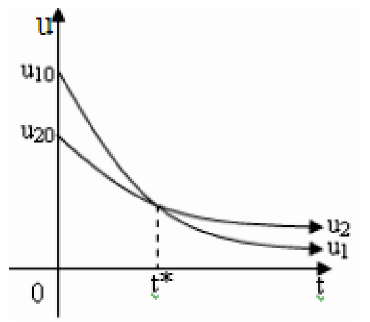
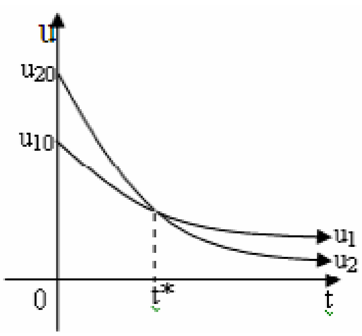
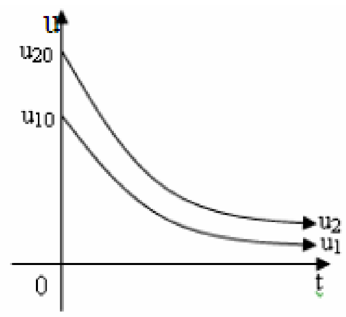
The above two roots λ_1 and λ_2 are negative and hence the steady state is **stable**.

The solutions of the linearized perturbed equations in this case are given by:

$$u_1 = u_{10} e^{-a_{11}\left(e_1 - \frac{c\left(k_2 + \sqrt{k_2^{\,2} - 4H_2}\right)}{2}\right)t} \qquad (4.16)$$

$$u_2 = u_{20} e^{-\left(a_{22}\sqrt{k_2^{\,2} - 4H_2}\right)t} \qquad (4.17)$$

The solution curves of (4.16) and (4.17) are shown in Figures 10 to 13 and the observations are presented in below.

	$a_{11}\left(e_1 - c\left(\frac{k_2 + \sqrt{k_2^{\,2} - 4H_2}}{2}\right)\right) < a_{22}\sqrt{k_2^{\,2} - 4H_2}$	$a_{11}\left(e_1 - c\left(\frac{k_2 + \sqrt{k_2^{\,2} - 4H_2}}{2}\right)\right) > a_{22}\sqrt{k_2^{\,2} - 4H_2}$
$u_{10} > u_{20}$	**Case-2A.1**  **Figure-10**	**Case-2A.2**  **Figure-11**
$u_{10} < u_{20}$	**Case-2A.3**  **Figure-12**	**Case-2A.4**  **Figure-13**

Observations:

Case-2A.1:

The initial population strength of the commensal is greater than that of the host i.e., $u_{10} > u_{20}$. In this case the commensal continues to out-number the host as shown in Figure-10. However both the species converge asymptotically to the equilibrium point.

Case-2A.2:

The initial population strength of the commensal is greater than that of the host i.e., $u_{10} > u_{20}$. Initially the commensal out-numbers the host and this continues up to the time

$$t^* = \frac{1}{a_{11}\left(e_1 - \frac{c\left(k_2 + \sqrt{k_2^2 - 4H_2}\right)}{2}\right) - a_{22}\sqrt{k_2^2 - 4H_2}} \log\left(\frac{u_{10}}{u_{20}}\right)$$

after which the host out-numbers the commensal. This is illustrated in Figure-11.

Case-2A.3:

The initial population strength of the host is greater than that of the commensal i.e., $u_{10} < u_{20}$. Initially the host out-numbers the commensal and this continues up to the time

$$t^* = \frac{1}{a_{22}\sqrt{k_2^2 - 4H_2} - a_{11}\left(e_1 - \frac{c\left(k_2 + \sqrt{k_2^2 - 4H_2}\right)}{2}\right)} \log\left(\frac{u_{20}}{u_{10}}\right)$$

after which the dominance is reversed as shown in Figure-12.

Case-2A.4:

The initial population strength of the host is greater than that of the commensal i.e., $u_{10} < u_{20}$. In this case the host continues to out-number the commensal as shown in Figure-13.

4.2 (a) Trajectories of the perturbed species

Eliminating 't' between the equations (4.16) and (4.17), we obtain:

$$\left(\frac{u_1}{u_{10}}\right) = \left(\frac{u_2}{u_{20}}\right)^{\gamma} \tag{4.18}$$

where $\gamma = \frac{a_{11}\left(e_1 - c\left(\frac{k_2 + \sqrt{k_2^2 - 4H_2}}{2}\right)\right)}{a_{22}\sqrt{k_2^2 - 4H_2}}$ and the resulting curves are parabolic type and are shown in Figure-14. This figure exhibit the stability of the equilibrium state.

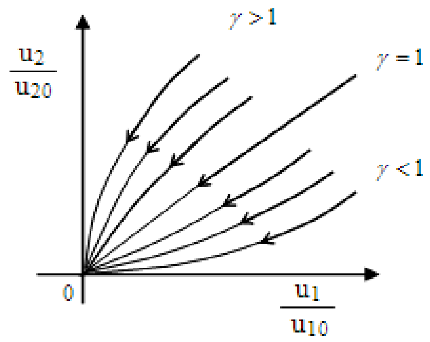


Figure-14

Case-2B: When $e_1 < c\left(\frac{k_2 + \sqrt{k_2^2 - 4H_2}}{2}\right)$

In this case one root (λ_1) of the above two roots is positive so the state is unstable.

The solutions of the linearized perturbed equations in this case are given by:

$$u_1 = u_{10} e^{a_{11}\left(\frac{c\left(k_2 + \sqrt{k_2^2 - 4H_2}\right)}{2} - e_1\right)t} \tag{4.19}$$

$$u_2 = u_{20} e^{-\left(a_{22}\sqrt{k_2^2 - 4H_2}\right)t} \tag{4.20}$$

and the solution curves are discussed as below.

Case-2B.1: When $u_{10} > u_{20}$

The commensal species always out-number the host species in natural growth rate as well as in its initial population strength, where as the host declines further is shown in Figure-15.

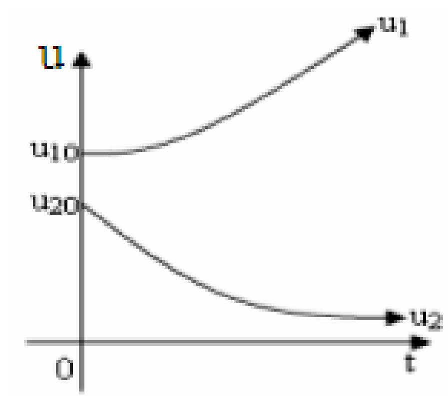


Figure-15

Case-2B.2: When $u_{10} < u_{20}$

The commensal dominates over the host in its natural growth rate but its initial strength is less than that of the host i.e., $u_{10} < u_{20}$. In this case, the host out-numbers the commensal till the time instant $t^* = \frac{1}{a_{11}\left(c\left[\frac{k_2+\sqrt{k_2^2-4H_2}}{2}\right]-e_1\right)+a_{22}\sqrt{k_2^2-4H_2}}\log\left(\frac{u_{20}}{u_{10}}\right)$ and there after the commensal out-numbers the host. This is seen in Figure-16.

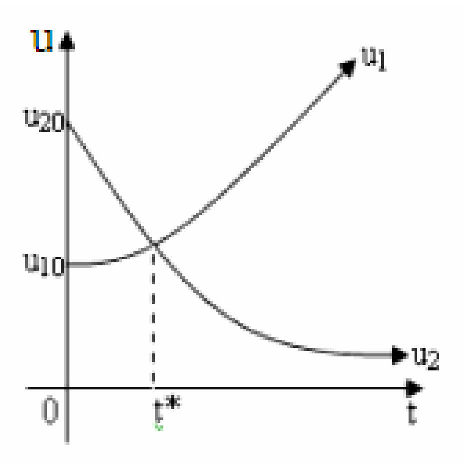


Figure-16

4.2 (b) Trajectories of the perturbed species

Eliminating‘t’ between the equations (4.19) and (4.20), we obtain:

$$\left(\frac{u_1}{u_{10}}\right)^{a_{22}\sqrt{k_2^2-4H_2}} = \left(\frac{u_2}{u_{20}}\right)^{-a_{11}\left(c\left(\frac{k_2+\sqrt{k_2^2-4H_2}}{2}\right)-e_1\right)} \quad (4.21)$$

and the trajectories are hyperbolic type as shown in Figure-17.

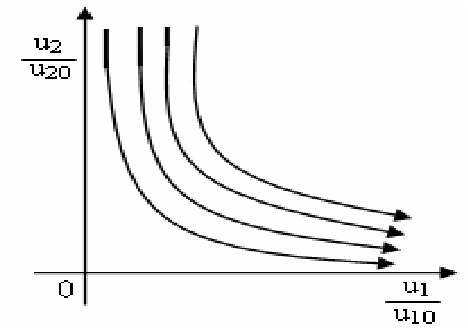


Figure-17

Case-2C: When $e_1 = c\left(\frac{k_2+\sqrt{k_2^2-4H_2}}{2}\right)$

In this case one root (λ_1) would be zero while the other root is negative and hence the state is unstable.

The solutions of the linearized perturbed equations in this case are given by:

$$u_1 = u_{10} \quad (4.22)$$

$$u_2 = u_{20}e^{-\left(a_{22}\sqrt{k_2^2-4H_2}\right)t} \quad (4.23)$$

The solution curves of (4.22) and (4.23) are illustrated in Figures 18 and 19.

Case-2C.1: When $u_{10} > u_{20}$

The initial population strength of the commensal is greater than that of the host i.e., $u_{10} > u_{20}$. In this case the host decays while the strength of the commensal remains constant, the death rate of which is compensated by the support given by the host. This is illustrated in Figure-18.

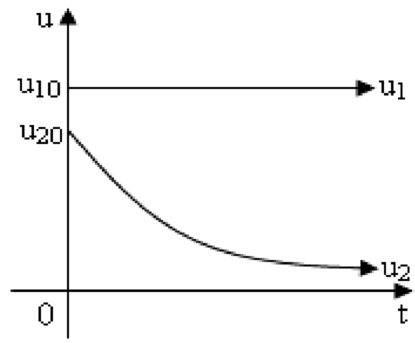


Figure-18

Case-2C.2: When $u_{10} < u_{20}$

The initial population strength of the host is greater than that of the commensal i.e., $u_{10} < u_{20}$. In this case the host dominates over the commensal and this continues up to the time instant $t = t^* = \frac{1}{a_{22}\sqrt{k_2^2-4H_2}}\log\left(\frac{u_{20}}{u_{10}}\right)$ after which the commensal dominates over the host. Further the commensal species diverge away from the equilibrium point while the host species is asymptotic to the equilibrium point. This is seen in Figure-19.

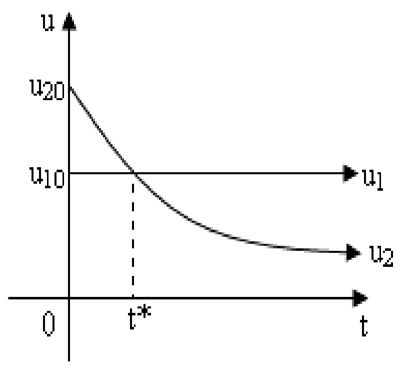


Figure-19

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4.2 (c) Trajectories of the perturbed species

Eliminating 't' between the equations (4.22) and (4.23), we obtain:

$$\frac{u_1}{u_{10}} = 1 \quad (4.24)$$

and the corresponding trajectory is a straight line shown in Figure-20.

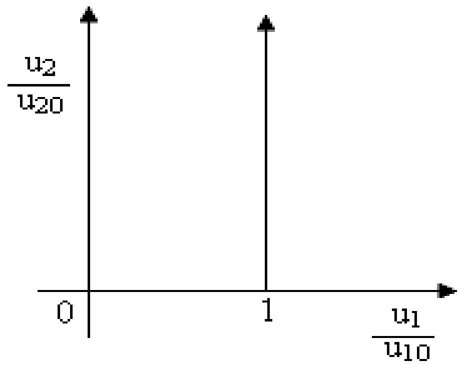

Figure-20

4.3. Stability of the equilibrium state

$$E_3 : \overline{N_1} = 0 \, ; \, \overline{N_2} = \frac{k_2 - \sqrt{k_2^{\,2} - 4H_2}}{2}$$

The corresponding characteristic matrix in this state is:

$$A = \begin{bmatrix} a_{11}\left[c\left(\frac{k_2 - \sqrt{k_2^{\,2} - 4H_2}}{2}\right) - e_1\right] & 0 \\ 0 & a_{22}\sqrt{k_2^{\,2} - 4H_2} \end{bmatrix}$$

The corresponding characteristic roots are:

$$\lambda_1 = a_{11}\left[c\left(\frac{k_2 - \sqrt{k_2^{\,2} - 4H_2}}{2}\right) - e_1\right]$$

and $\lambda_2 = a_{22}\sqrt{k_2^{\,2} - 4H_2} > 0$.

Since one of the two roots is positive, this state is unstable.

As before three cases will arise:

Case-3A: When $e_1 > c\left(\frac{k_2 - \sqrt{k_2^{\,2} - 4H_2}}{2}\right)$

The solutions of the linearized perturbed equations in this case are given by:

$$u_1 = u_{10}e^{-a_{11}\left(e_1 - \frac{c\left(k_2 - \sqrt{k_2^{\,2} - 4H_2}\right)}{2}\right)t} \quad (4.25)$$

$$u_2 = u_{20}e^{\left(a_{22}\sqrt{k_2^{\,2} - 4H_2}\right)t} \quad (4.26)$$

The solution curves of (4.25) and (4.26) are shown in the following figures and the observations are presented in below.

Case-3A.1: When $u_{10} > u_{20}$

The host dominates over the commensal in its natural growth rate but its initial strength is less than that of the commensal i.e., $u_{10} > u_{20}$. In this case, the commensal out-numbers the host till the time instant

$$t^* = \frac{1}{a_{11}\left(c\left(\frac{k_2 - \sqrt{k_2^{\,2} - 4H_2}}{2}\right) - e_1\right) + a_{22}\sqrt{k_2^{\,2} - 4H_2}} \log\left(\frac{u_{10}}{u_{20}}\right)$$ and

there after the dominance is reversed is shown in Figure-21.

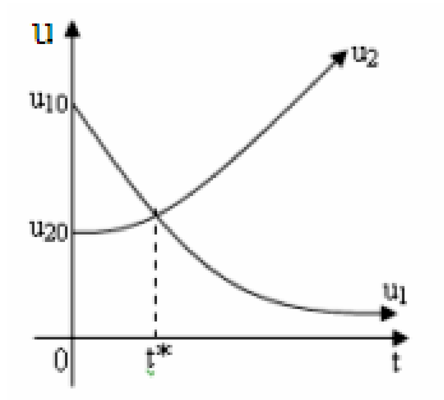

Figure-21

Case-3A.2: When $u_{10} < u_{20}$

The host species always out-number the commensal species in natural growth rate as well as in its initial population strength, where as the commensal declines further as shown in Figure-22.

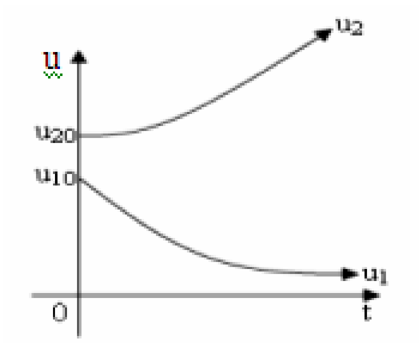

Figure-22

4.3 (a) Trajectories of the perturbed species

Eliminating 't' between the equations (4.25) and (4.26), we obtain:

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$$\left(\frac{u_1}{u_{10}}\right)^{a_{22}\sqrt{k_2^2-4H_2}} = \left(\frac{u_2}{u_{20}}\right)^{-a_{11}\left(e_1-c\left(\frac{k_2-\sqrt{k_2^2-4H_2}}{2}\right)\right)} \qquad (4.27)$$

and the trajectories are hyperbolic type as shown in Figure-23.

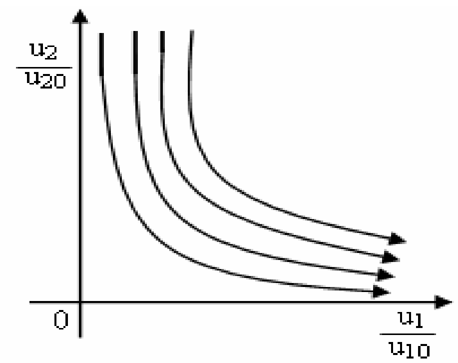


Figure-23

Case-3B: When $e_1 < c\left(\dfrac{k_2-\sqrt{k_2^2-4H_2}}{2}\right)$

In this case the solutions of the linearized perturbed equations are given by:

$$u_1 = u_{10}e^{a_{11}\left(\frac{c\left(k_2-\sqrt{k_2^2-4H_2}\right)}{2}-e_1\right)t} \qquad (4.28)$$

$$u_2 = u_{20}e^{\left(a_{22}\sqrt{k_2^2-4H_2}\right)t} \qquad (4.29)$$

The solution curves in this case are illustrated below from Figures 24 to 27.

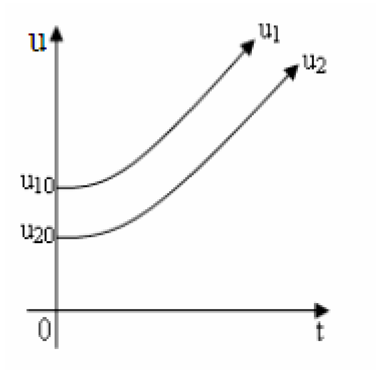
<table>
<tr><td></td><td>$a_{11}\left(c\left(\frac{k_2-\sqrt{k_2^2-4H_2}}{2}\right)-e_1\right) > a_{22}\sqrt{k_2^2-4H_2}$</td><td>$a_{11}\left(c\left(\frac{k_2-\sqrt{k_2^2-4H_2}}{2}\right)-e_1\right) < a_{22}\sqrt{k_2^2-4H_2}$</td></tr>
<tr><td>$u_{10} > u_{20}$</td><td>Case-3B.1


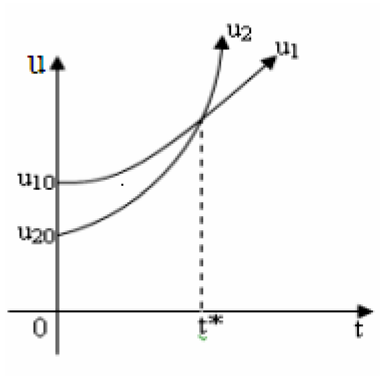
Figure-24</td><td>Case-3B.2


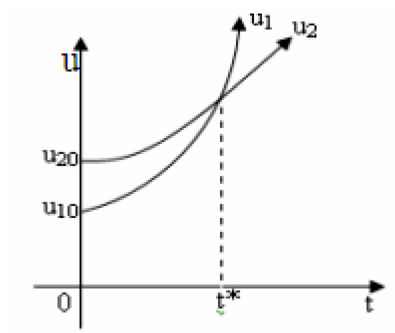
Figure-25</td></tr>
<tr><td>$u_{10} < u_{20}$</td><td>Case-3B.3


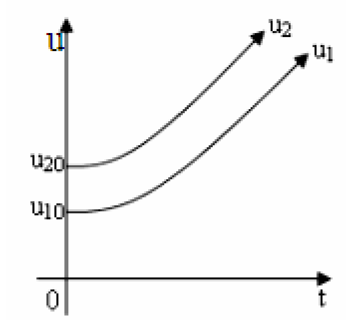
Figure-26</td><td>Case-3B.4


Figure-27</td></tr>
</table>

Observations:

Case-3B.1:

Initially, the first species out-numbers the second species and it continues to grow. Also we observe that both the species diverge away from the equilibrium point. Hence the equilibrium point is unstable as shown in Figure-24.

Case-3B.2:

The initial population strength of the commensal is greater than that of the host i.e., $u_{10} > u_{20}$. Initially, the commensal out-numbers the host and this continues up to the time

$$t^* = \frac{1}{a_{22}\sqrt{k_2^2 - 4H_2} - a_{11}\left(c\left[\frac{k_2 - \sqrt{k_2^2 - 4H_2}}{2}\right] - e_1\right)} \log\left(\frac{u_{10}}{u_{20}}\right)$$

after which, the host out-numbers the commensal as shown in Figure.25.

Case-3B.3:

The initial population strength of the host is greater than that of the commensal i.e., $u_{10} < u_{20}$. Initially, the host out-numbers the commensal and this continues up to the time

$$t^* = \frac{1}{a_{11}\left(c\left[\frac{k_2 - \sqrt{k_2^2 - 4H_2}}{2}\right] - e_1\right) - a_{22}\sqrt{k_2^2 - 4H_2}} \log\left(\frac{u_{20}}{u_{10}}\right)$$

after which, the dominance is reversed. This is illustrated in Figure-26.

Case-3B.4:

The initial population strength of the host is greater than that of the commensal i.e., $u_{10} < u_{20}$. In this case the second species out-number the first species all the time as shown in Figure-27.

4.3 (b) Trajectories of the perturbed species

Eliminating‘t’ between the equations (4.28) and (4.29), we obtain:

$$\left(\frac{u_1}{u_{10}}\right) = \left(\frac{u_2}{u_{20}}\right)^{\gamma} \qquad (4.30)$$

where $\gamma = \frac{a_{11}\left(c\left(\frac{k_2 - \sqrt{k_2^2 - 4H_2}}{2}\right) - e_1\right)}{a_{22}\sqrt{k_2^2 - 4H_2}}$ and the resulting curves are shown in Figure-28.

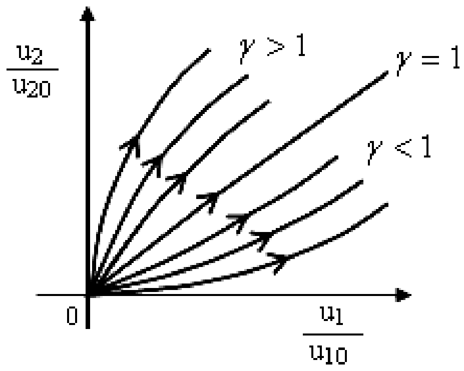


Figure-28

Case-3C: When $e_1 = c\left(\frac{k_2 - \sqrt{k_2^2 - 4H_2}}{2}\right)$

The solutions of the linearized perturbed equations in this case are given by:

$$u_1 = u_{10} \qquad (4.31)$$

$$u_2 = u_{20} e^{\left(a_{22}\sqrt{k_2^2 - 4H_2}\right)t} \qquad (4.32)$$

The solution curves of (4.31) and (4.32) are illustrated below.

Case-3C.1: When $u_{10} > u_{20}$

Initially the commensal dominates the host. In this case $u_1(t) = u_2(t)$ is possible at a time $t^* = \frac{1}{a_{22}\sqrt{k_2^2 - 4H_2}} \log\left(\frac{u_{10}}{u_{20}}\right)$, which is the dominance reversal time of the host species as shown in Figure-29.

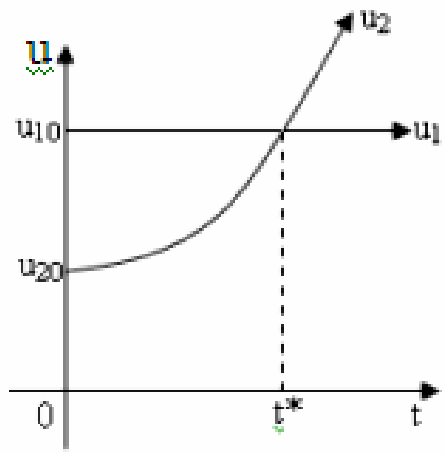


Figure-29

Case-3C.2: When $u_{10} < u_{20}$

In this case the host always out-numbers the commensal. Here the commensal species is observed at constant distance from the equilibrium point, while the host species diverge away from the equilibrium point is shown in Figure-30.

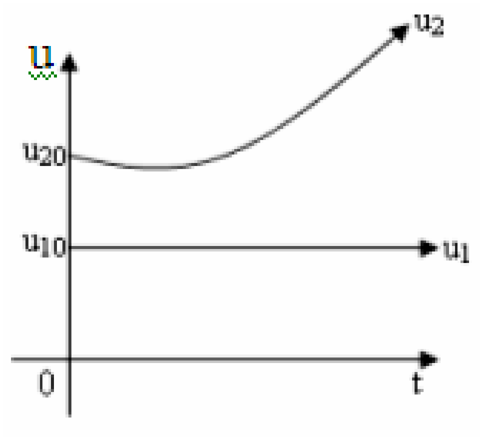


Figure-30

4.3 (c) Trajectories of the perturbed species

Eliminating'‘t’ between the equations (4.31) and (4.32), we obtain:

$$\frac{u_1}{u_{10}} = 1 \tag{4.33}$$

and the corresponding trajectory is a straight line as shown in Figure-31.

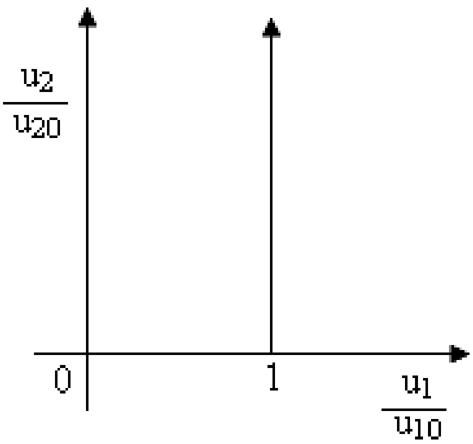


Figure-31

4.4. Stability of the equilibrium state

$E_4 : \overline{N_1} = \dfrac{ck_2 - 2e_1}{2}$; $\overline{N_2} = \dfrac{k_2}{2}$

The corresponding characteristic equation of this state is:

$$\lambda\left(\lambda + a_{11}\left(\frac{ck_2}{2} - e_1\right)\right) = 0 \tag{4.34}$$

The characteristic roots of the equation (4.34) are $\lambda_1 = -a_{11}\left(\dfrac{ck_2}{2} - e_1\right) < 0$ and $\lambda_2 = 0$. Since one root of the two roots would be zero so this state is unstable.

The solutions of the linearized perturbed equations in this state are:

$$u_1 = [u_{10} - L_1]e^{-a_{11}\left(\frac{ck_2}{2} - e_1\right)t} + L_1 \tag{4.35}$$

where $L_1 = cu_{20}$ (4.35.1)

$$u_2 = u_{20} \tag{4.36}$$

Two cases would arise here.

Case-4A: $u_{10} = L_1$**; Case-4B:** $u_{10} \neq L_1$

The solution curves in these two cases are illustrated below.

Case-4A.1: When $u_{10} > u_{20}$

The initial population strength of the commensal is greater than that of the host. In this case the commensal always out-numbers the host. Further both the species are at a constant distance from the equilibrium point as shown in Figure-32.

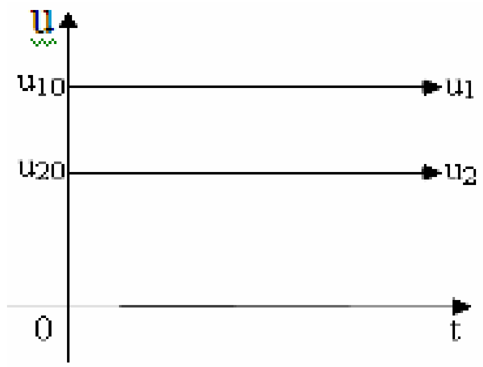


Figure-32

Case-4B.1: When $u_{10} > u_{20}$

In this case the commensal out-numbers the host till the time instant $t^* = \dfrac{2}{a_{11}(ck_2 - 2e_1)} \log\left(\dfrac{u_{10} - L_1}{u_{20} - L_1}\right)$ and there after the dominance is reversed. This is shown in Figure-33. Here the commensal is asymptotic to the equilibrium point while, the host goes far away from the equilibrium point.

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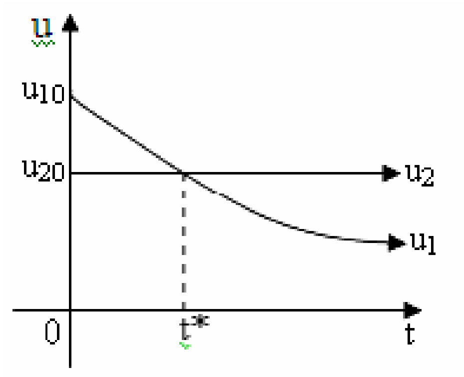

Figure-33

Case-4B.2: When $u_{10} < u_{20}$

The host continues to out-number the commensal in natural growth rate as well as in its initial population strength as shown in Figure-34.

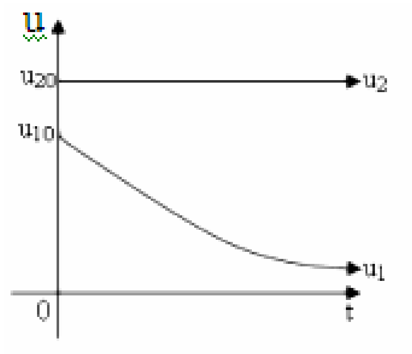

Figure-34

4.4 (a) Trajectories of the perturbed species

Eliminating 't' between the equations (4.35) and (4.36), we obtain:

$$\frac{u_2}{u_{20}} = 1 \tag{4.37}$$

and the resulting curve is a straight line as shown in Figure-35.

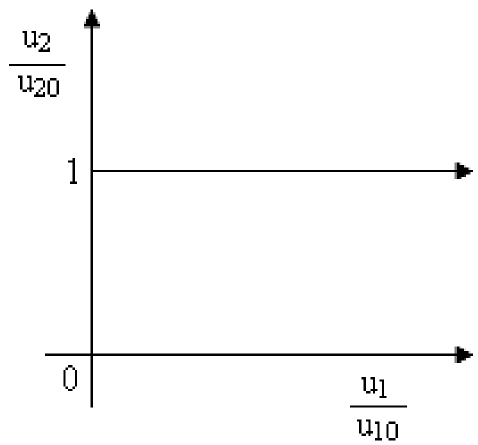

Figure-35

4.5. Stability of the equilibrium state E_5:

The corresponding linearized perturbed equations are:

$$\frac{d}{dt}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -a_{11}\left[c\left(\frac{k_2+\sqrt{k_2^{\,2}-4H_2}}{2}\right)-e_1\right] & ca_{11}\left[c\left(\frac{k_2+\sqrt{k_2^{\,2}-4H_2}}{2}\right)-e_1\right] \\ 0 & -a_{22}\sqrt{k_2^{\,2}-4H_2} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{4.38}$$

The corresponding characteristic roots are $\lambda_1 = -a_{11}\left[\frac{c\left[k_2+\sqrt{k_2^{\,2}-4H_2}\right]}{2} - e_1\right] < 0$ and $\lambda_2 = -a_{22}\sqrt{k_2^{\,2}-4H_2} < 0$ both negative, the steady state is **stable**.

The solutions of the equations in (4.38) are given by:

$$u_1 = \left[u_{10} - L_2\right]e^{-a_{11}\left(c\left(\frac{k_2+\sqrt{k_2^{\,2}-4H_2}}{2}\right)-e_1\right)t} + L_2 e^{-\left(a_{22}\sqrt{k_2^{\,2}-4H_2}\right)t} \tag{4.39}$$

where

$$L_2 = \frac{a_{11}cu_{20}\left[\frac{c\left[k_2+\sqrt{k_2^{\,2}-4H_2}\right]}{2}-e_1\right]}{a_{11}\left[\frac{c\left(k_2+\sqrt{k_2^{\,2}-4H_2}\right)}{2}-e_1\right]-a_{22}\sqrt{k_2^{\,2}-4H_2}} \quad (4.39.1)$$

$$u_2 = u_{20}e^{-\left(a_{22}\sqrt{k_2^{\,2}-4H_2}\right)t} \quad (4.40)$$

It is to be noted that $\frac{c\left[k_2+\sqrt{k_2^{\,2}-4H_2}\right]}{2} > e_1$ and also noticed that $(u_1,u_2)\to 0$ as $t\to\infty$.
There arise the following two cases:

Case-5A: $u_{10} = L_2$ **; Case-5B:** $u_{10} \neq L_2$

Case-5A: When $u_{10} = L_2$

the equations (4.39) and (4.40) become:

$$u_1 = u_{10}e^{-\left(a_{22}\sqrt{k_2^{\,2}-4H_2}\right)t} \quad (4.41)$$

$$u_2 = u_{20}e^{-\left(a_{22}\sqrt{k_2^{\,2}-4H_2}\right)t} \quad (4.42)$$

Here both u_1 and u_2 are exponentially decay with the same characteristic time $1/a_{22}\sqrt{k_2^{\,2}-4H_2}$, the initial values ($u_{10}$ and u_{20}) may however be different. Hence the equilibrium point is stable.
The solution curves in this case are given as follows:

Case-5A.1: When $u_{10} > u_{20}$

In this case the commensal species always out-number the host species in natural growth rate as well as in its initial population strength. It is noted that both the commensal and the host converge asymptotically to the equilibrium point as shown in Figure-36.

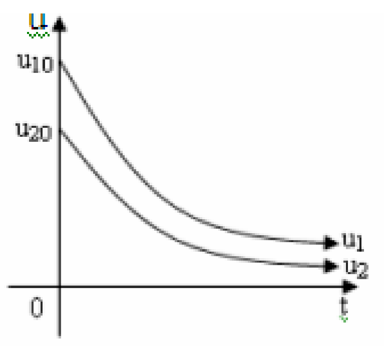

Figure-36

Case-5A.2: When $u_{10} < u_{20}$

The host species dominates over the commensal species in its initial population strength. Also both the species move towards to the equilibrium point as seen in Figure-37.

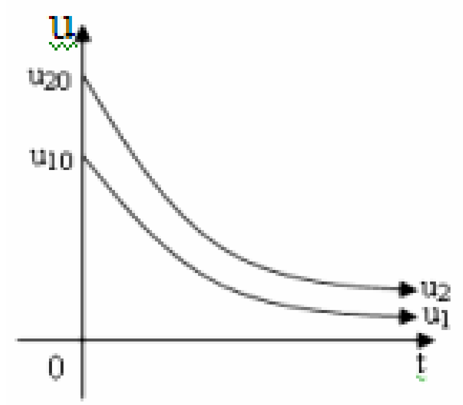

Figure-37

4.5 (a) Trajectories of the perturbed species

Eliminating 't' between the equations (4.41) and (4.42), we obtain:

$$\frac{u_1}{u_{10}} = \frac{u_2}{u_{20}} \quad (4.43)$$

and the corresponding trajectory is a straight line as shown in Figure-38.

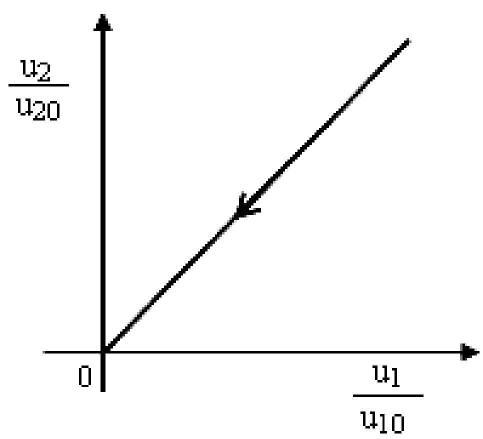

Figure-38

Case-5B: $u_{10} \neq L_2$

Case-5B.1: When $u_{10} > u_{20}$ **and**

$$a_{11}\left[\frac{c\left[k_2+\sqrt{k_2^{\,2}-4H_2}\right]}{2}-e_1\right] > a_{22}\sqrt{k_2^{\,2}-4H_2}$$

The initial population strength of the commensal is greater than that of the host i.e., $u_{10} > u_{20}$. In this case the commensal dominates over the host till the time instant

$$t^* = \frac{1}{a_{11}\left(\frac{c\left(k_2+\sqrt{k_2^{\,2}-4H_2}\right)}{2}-e_1\right)-a_{22}\sqrt{k_2^{\,2}-4H_2}}\log\left(\frac{u_{10}-L_2}{u_{20}-L_2}\right)$$

after which the host dominates. This is the dominance reversal time in this case as shown in Figure-39.

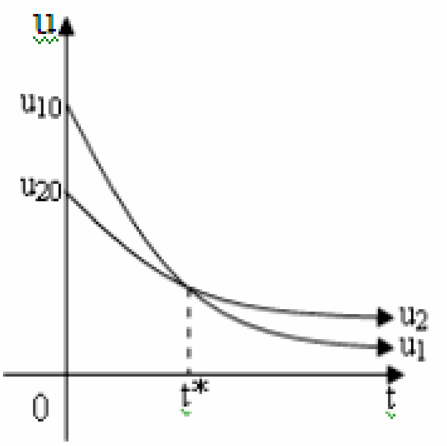

Figure-39

Case-5B.2: When $u_{10} > u_{20}$ and

$$a_{11}\left[\frac{c\left[k_2+\sqrt{k_2^{\,2}-4H_2}\right]}{2}-e_1\right] < a_{22}\sqrt{k_2^{\,2}-4H_2}$$

The initial population strength of the commensal is greater than that of the host i.e., $u_{10} > u_{20}$. In this case the commensal dominates the host all the time as shown in Figure-40.

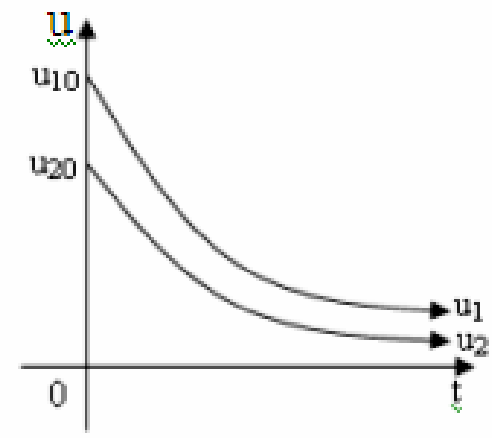

Figure-40

Case-5B.3: When $u_{10} < u_{20}$ and

$$a_{11}\left[\frac{c\left[k_2+\sqrt{k_2^{\,2}-4H_2}\right]}{2}-e_1\right] > a_{22}\sqrt{k_2^{\,2}-4H_2}$$

The initial population strength of the host is greater than that of the commensal i.e., $u_{10} < u_{20}$. In this case the host continues to out-number the commensal as shown in Figure-41.

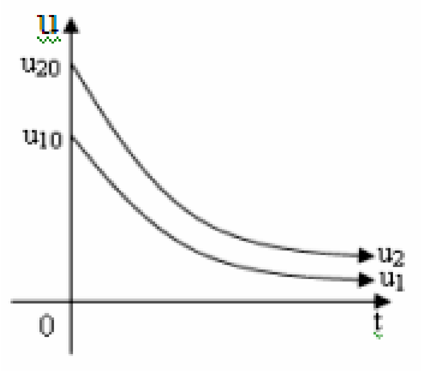

Figure-41

Case-5B.4: When $u_{10} < u_{20}$ and

$$a_{11}\left[\frac{c\left[k_2+\sqrt{k_2^{\,2}-4H_2}\right]}{2}-e_1\right] < a_{22}\sqrt{k_2^{\,2}-4H_2}$$

The initial population strength of the host is greater than that of the commensal i.e., $u_{10} < u_{20}$. In this case initially the host out- numbers the commensal and this continues up to the time instant

$$t^* = \frac{1}{a_{22}\sqrt{k_2^{\,2}-4H_2}-a_{11}\left(\frac{c\left(k_2+\sqrt{k_2^{\,2}-4H_2}\right)}{2}-e_1\right)}\log\left(\frac{u_{20}+L_2'}{u_{10}+L_2'}\right)$$

where $L_2' = -L_2$ after which, the dominance is reversed. The dominance reversal time is shown in Figure-42.

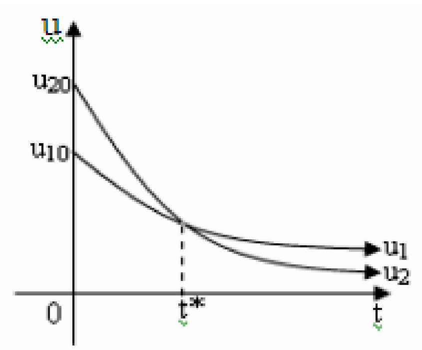

Figure-42

4.5 (b) Trajectories of the perturbed species

Eliminating 't' between the equations (4.39) and (4.40), we obtain:

$$\frac{u_1}{u_{10}} = \left(\frac{L_2}{u_{10}}\right)\left(\frac{u_2}{u_{20}}\right) + \left(1 - \frac{L_2}{u_{10}}\right)\left(\frac{u_2}{u_{20}}\right)^{\gamma} \quad (4.44)$$

where $\gamma = \dfrac{a_{11}\left[c\left(\dfrac{k_2 + \sqrt{k_2^2 - 4H_2}}{2}\right) - e_1\right]}{a_{22}\sqrt{k_2^2 - 4H_2}}$ and the resulting curves are parabolic type and are shown in Figure-43. This figure exhibits the stability of the equilibrium state.

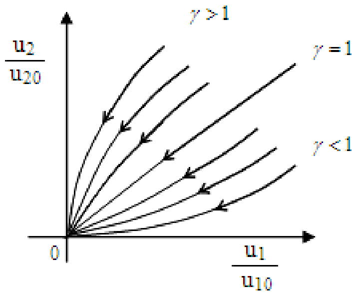

Figure-43

4.6. Stability of the equilibrium state E_6 :

$$\overline{N_1} = \frac{c\left[k_2 - \sqrt{k_2^2 - 4H_2}\right]}{2} - e_1 ;\ \overline{N_2} = \frac{k_2 - \sqrt{k_2^2 - 4H_2}}{2}$$

along with $e_1 < \dfrac{c\left[k_2 - \sqrt{k_2^2 - 4H_2}\right]}{2}$ and the corresponding characteristic matrix is:

$$A = \begin{bmatrix} -a_{11}\left[c\left(\dfrac{k_2 - \sqrt{k_2^2 - 4H_2}}{2}\right) - e_1\right] & ca_{11}\left[c\left(\dfrac{k_2 - \sqrt{k_2^2 - 4H_2}}{2}\right) - e_1\right] \\ 0 & a_{22}\sqrt{k_2^2 - 4H_2} \end{bmatrix} \quad (4.45)$$

The characteristic equation of (4.45) is:

$$\left(\lambda + a_{11}\left[c\left(\frac{k_2 - \sqrt{k_2^2 - 4H_2}}{2}\right) - e_1\right]\right)\left(\lambda - a_{22}\sqrt{k_2^2 - 4H_2}\right) = 0 \quad (4.46)$$

The characteristic roots of (4.46) are:

$$\lambda_1 = -a_{11}\left[\frac{c\left[k_2 - \sqrt{k_2^2 - 4H_2}\right]}{2} - e_1\right] < 0$$ and

$$\lambda_2 = a_{22}\sqrt{k_2^2 - 4H_2} > 0 .$$

Since one of the two roots is positive then the steady state is **unstable**.

In this state the solutions of linearized perturbed equations are as follows:

$$u_1 = [u_{10} - L_3]e^{-a_{11}\left(c\left(\frac{k_2 - \sqrt{k_2^2 - 4H_2}}{2}\right) - e_1\right)t} + L_3 e^{\left(a_{22}\sqrt{k_2^2 - 4H_2}\right)t} \quad (4.47)$$

where
$$L_3 = \frac{a_{11}cu_{20}\left[\dfrac{c\left[k_2 - \sqrt{k_2^2 - 4H_2}\right]}{2} - e_1\right]}{a_{11}\left[c\left[\dfrac{k_2 - \sqrt{k_2^2 - 4H_2}}{2}\right] - e_1\right] + a_{22}\sqrt{k_2^2 - 4H_2}} \quad (4.47.1)$$

$$u_2 = u_{20}e^{\left(a_{22}\sqrt{k_2^2 - 4H_2}\right)t} \quad (4.48)$$

There arise the following **two** cases.

Case-6A: $u_{10} = L_3$ **; Case-6B:** $u_{10} \neq L_3$

The solution curves in these cases are illustrated as follows:

Case-6A: When $u_{10} = L_3$

the equations (4.47) and (4.48) become:

$$u_1 = u_{10}e^{\left(a_{22}\sqrt{k_2^2 - 4H_2}\right)t} \quad (4.49)$$

$$u_2 = u_{20}e^{\left(a_{22}\sqrt{k_2^2 - 4H_2}\right)t} \quad (4.50)$$

Case-6A.1: When $u_{10} > u_{20}$

The initial population strength of the commensal is greater than that of the host. However, both the species move away from the equilibrium point. This is illustrated in Figure- 44.

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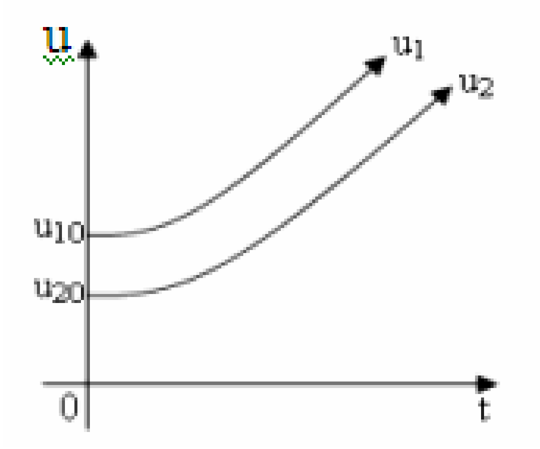

Figure-44

Case-6A.2: When $u_{10} < u_{20}$

The initial population strength of the host is greater than that of the commensal i.e., $u_{10} < u_{20}$. In this case the host dominates the commensal all the time as shown in Figure-45.

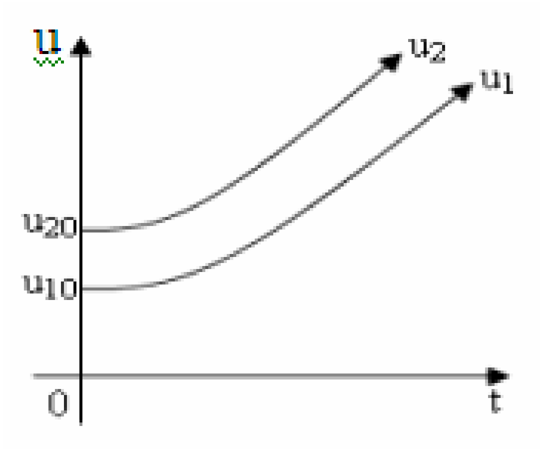

Figure-45

4.6 (a) Trajectories of the perturbed species

Eliminating't' between the equations (4.49) and (4.50), we obtain:

$$\frac{u_1}{u_{10}} = \frac{u_2}{u_{20}} \tag{4.51}$$

and the corresponding trajectory is a straight line as shown in Figure-46.

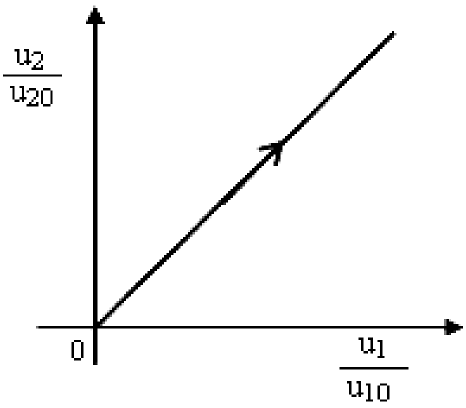

Figure-46

Case-6B: $u_{10} \neq L_3$

Case-6B.1: When $u_{10} > u_{20}$

The host dominates over the commensal after the time instant t*, but its initial population strength is less than that of the commensal. Here, the host dominance time over the commensal is

$$t^* = \frac{1}{a_{11}\left(c\left(\frac{k_2 - \sqrt{k_2^{\,2} - 4H_2}}{2}\right) - e_1\right) + a_{22}\sqrt{k_2^{\,2} - 4H_2}} \log\left(\frac{u_{10} - L_3}{u_{20} - L_3}\right)$$

This is illustrated in Figure-47.

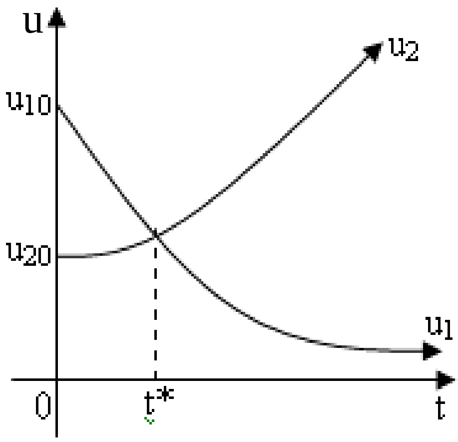

Figure-47

Case-6B.2: When $u_{10} < u_{20}$

In this case the host species always out-number the commensal species. Also it is evident that the host species goes far away from the equilibrium point while the commensal is asymptotic to the equilibrium point as shown in Figure-48.

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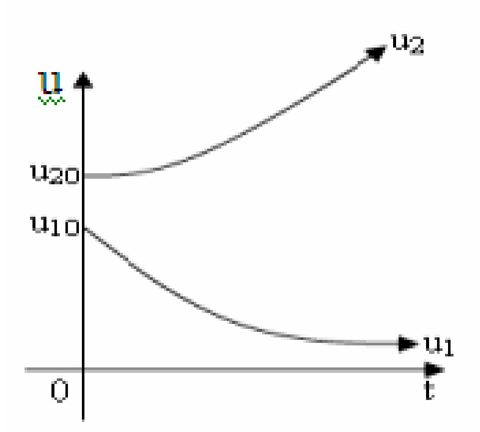

Figure-48

4.6 (b) Trajectories of the perturbed species

Eliminating't' between the equations (4.47) and (4.48), we obtain:

$$\frac{u_1}{u_{10}} = \left(\frac{L_3}{u_{10}}\right)\left(\frac{u_2}{u_{20}}\right) + \left(1 - \frac{L_3}{u_{10}}\right)\left(\frac{u_2}{u_{20}}\right)^{-\gamma} \quad (4.52)$$

where $\gamma = \dfrac{a_{11}\left[c\left(\dfrac{k_2 - \sqrt{k_2^2 - 4H_2}}{2}\right) - e_1\right]}{a_{22}\sqrt{k_2^2 - 4H_2}}$ the resulting curves are shown in Figure-49.

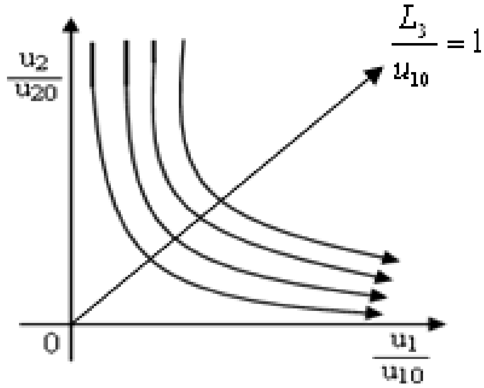

Figure-49

5. THRESHOLD (OR) PHASE - PLANE DIAGRAM

The conditions $\frac{dN_1}{dt} = 0$ and $\frac{dN_2}{dt} = 0$ imply that neither N_1 nor N_2 changes its density. When we impose these conditions the basic equations give rise to four straight lines. At the points where $\frac{dN_1}{dt} = 0$; $\frac{dN_2}{dt} = 0$, the resulting straight lines divide the phase plane in to eight regions in the first quadrant $N_1 \geq 0, N_2 \geq 0$ (vide Figure-50).

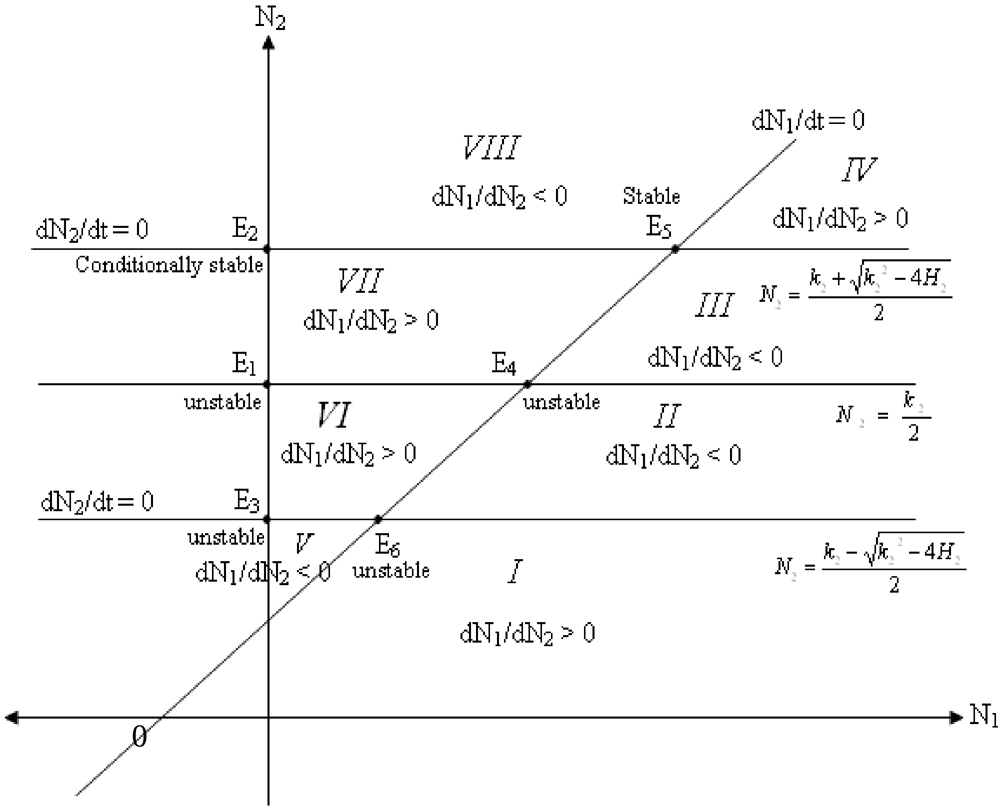

Figure-50

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Threshold regions

Region I: Both the species N_1 and N_2 decline with time t.

Region II: The commensal species N_1 declines and the host species N_2 flourishes with time t.

Region III: The commensal species N_1 declines and the host species N_2 flourishes with time t.

Region IV: Both the species N_1 and N_2 decline with time t.

Region V: The commensal species N_1 flourishes and the host species N_2 declines with time t.

Region VI: Both the species N_1 and N_2 flourish with time t.

Region VII: Both the species N_1 and N_2 flourish with time t.

Region VIII: Both the species N_1 and N_2 decline with time t.

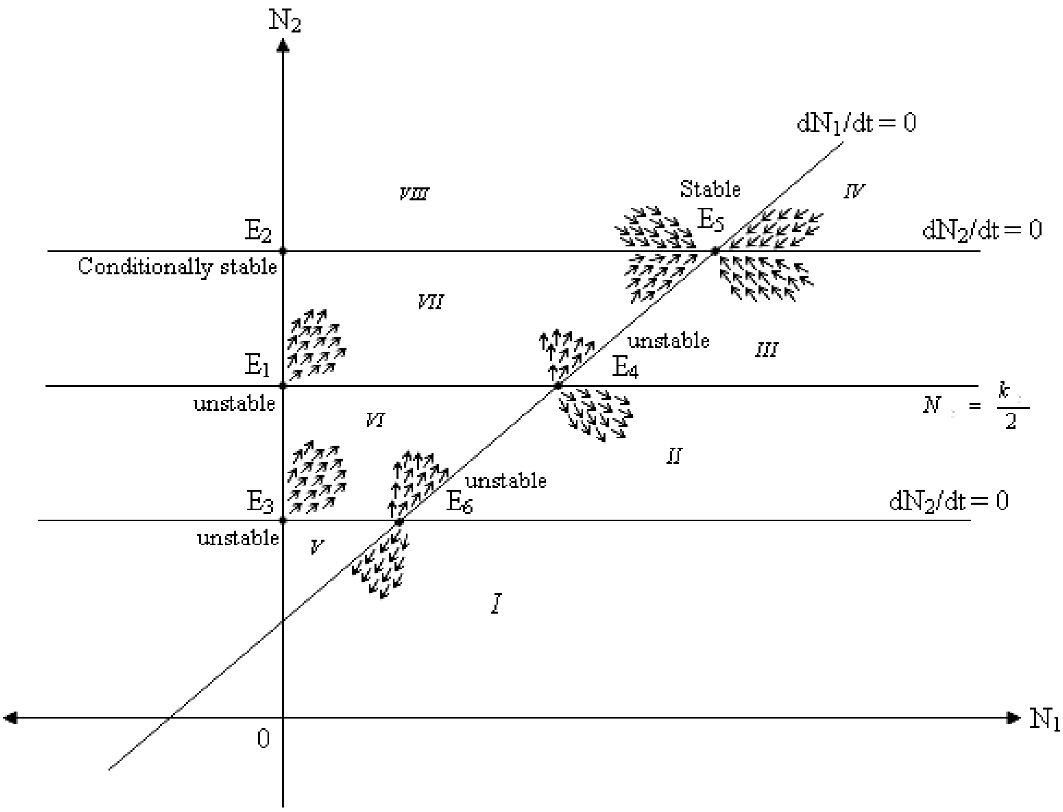

Figure-51 (Threshold diagram)

6. LIAPUNOV'S FUNCTION FOR GLOBAL STABILITY

In Section 4.5 we have discussed the local stability of the state of co-existence. Now we examine the global stability of the dynamical system (2.1) and (2.2). We have already noted that this system has a unique, stable non-trivial co-existent equilibrium state at

$$\overline{N_1} = \frac{c\left[k_2 + \sqrt{k_2^{\,2} - 4H_2}\right]}{2} - e_1;\ \overline{N_2} = \frac{k_2 + \sqrt{k_2^{\,2} - 4H_2}}{2}.$$

Basic equations:

$$\frac{dN_1}{dt} = a_{11} N_1 \left[-e_1 - N_1 + cN_2\right] \quad (6.1)$$

$$\frac{dN_2}{dt} = a_{22}\left[k_2 N_2 - N_2^{\,2} - H_2\right] \quad (6.2)$$

The linearized perturbed equations over the perturbations (u_1, u_2) are:

$$\frac{du_1}{dt} = -a_{11}\overline{N_1} u_1 + c a_{11}\overline{N_1} u_2 \quad (6.3)$$

$$\frac{du_2}{dt} = -2a_{22}\left[\overline{N_2} - \frac{k_2}{2}\right] u_2 \quad (6.4)$$

The corresponding characteristic equation is:

$$\left(\lambda + a_{11}\overline{N_1}\right)\left(\lambda + 2a_{22}\left[\overline{N_2} - \frac{k_2}{2}\right]\right) = 0 \quad (6.5)$$

$$\lambda^2 + \left[a_{11}\overline{N_1} + 2a_{22}\left[\overline{N_2} - \frac{k_2}{2}\right]\right]\lambda + 2a_{11}a_{22}\overline{N_1}\left[\overline{N_2} - \frac{k_2}{2}\right] = 0 \quad (6.6)$$

Equation (6.6) is of the form $\lambda^2 + p\lambda + q = 0$

where

$$p = a_{11}\overline{N_1} + 2a_{22}\left[\overline{N_2} - \frac{k_2}{2}\right] > 0 \quad (6.7)$$

$$q = 2a_{11}a_{22}\overline{N_1}\left[\overline{N_2} - \frac{k_2}{2}\right] > 0 \quad (6.8)$$

$\therefore$ The conditions for the existence of Liapunov's function are satisfied.

Now define

$$E\,(u_1, u_2) = \frac{1}{2}\left(Au_1^{\,2} + 2Bu_1u_2 + Cu_2^{\,2}\right) \quad (6.9)$$

where

$$A = \frac{\left(2a_{22}\left[\overline{N_2} - \frac{k_2}{2}\right]\right)^2 + 2a_{11}a_{22}\overline{N_1}\left[\overline{N_2} - \frac{k_2}{2}\right]}{D} \quad (6.10)$$

$$B = \frac{2ca_{11}a_{22}\overline{N_1}\left[\overline{N_2} - \frac{k_2}{2}\right]}{D} \quad (6.11)$$

$$C = \frac{a_{11}^{\,2}\overline{N_1}^{\,2} + \left(ca_{11}\overline{N_1}\right)^2 + 2a_{11}a_{22}\overline{N_1}\left[\overline{N_2} - \frac{k_2}{2}\right]}{D} \quad (6.12)$$

And

$$D = pq > 0 \quad (6.13)$$

From the equations (6.7) and (6.8) it is clear that D>0. Also

$$D^2\left(AC - B^2\right) = \left\{\left(\left(2a_{22}\left[\overline{N_2} - \frac{k_2}{2}\right]\right)^2 + 2a_{11}a_{22}\overline{N_1}\left[\overline{N_2} - \frac{k_2}{2}\right]\right)\left(a_{11}^{\,2}\overline{N_1}^{\,2} + \left(ca_{11}\overline{N_1}\right)^2 + 2a_{11}a_{22}\overline{N_1}\left[\overline{N_2} - \frac{k_2}{2}\right]\right) - \left(2ca_{11}a_{22}\overline{N_1}\right)^2\left[\overline{N_2} - \frac{k_2}{2}\right]^2\right\}$$

$$\Rightarrow D^2\left(AC - B^2\right) > 0 \Rightarrow AC - B^2 > 0 \text{ i.e., } B^2 - AC < 0 \quad (6.14)$$

$\therefore$ The function E (u_1, u_2) at (6.9) is positive definite.

Further $$\frac{\partial E}{\partial u_1}\frac{du_1}{dt} + \frac{\partial E}{\partial u_2}\frac{du_2}{dt} = \left(Au_1 + Bu_2\right)\left(-a_{11}\overline{N_1}u_1 + ca_{11}\overline{N_1}u_2\right) + \left(Bu_1 + Cu_2\right)\left(-2a_{22}\left[\overline{N_2} - \frac{k_2}{2}\right]u_2\right)$$
(6.15)

Substituting the values of A, B and C from (6.10) (6.11) and (6.12) in (6.15) we get

$$\frac{\partial E}{\partial u_1}\frac{du_1}{dt} + \frac{\partial E}{\partial u_2}\frac{du_2}{dt} = -\frac{1}{D}\left[a_{11}\overline{N_1}\left[\left(2a_{22}\left[\overline{N_2} - \frac{k_2}{2}\right]\right)^2 + 2a_{11}a_{22}\overline{N_1}\left[\overline{N_2} - \frac{k_2}{2}\right]\right]\right] u_1^2$$

$$+\frac{1}{D}\left[ca_{11}\overline{N_1}\left\{\left(2a_{22}\left[\overline{N_2} - \frac{k_2}{2}\right]\right)^2 + 2a_{11}a_{22}\overline{N_1}\left[\overline{N_2} - \frac{k_2}{2}\right]\right\} - a_{11}c\overline{N_1}\left(2a_{11}a_{22}\overline{N_1}\left[\overline{N_2} - \frac{k_2}{2}\right]\right) - 4ca_{11}a_{22}^{\,2}\overline{N_1}\left[\overline{N_2} - \frac{k_2}{2}\right]^2\right] u_1u_2$$

$$-\frac{1}{D}\left\{ca_{11}\overline{N_1}\left(2ca_{11}a_{22}\overline{N_1}\left[\overline{N_2} - \frac{k_2}{2}\right]\right) - 2a_{22}\left[\overline{N_2} - \frac{k_2}{2}\right]\left[\left(a_{11}\overline{N_1}\right)^2 + \left(ca_{11}\overline{N_1}\right)^2 + \left(2a_{11}a_{22}\overline{N_1}\left[\overline{N_2} - \frac{k_2}{2}\right]\right)\right]\right\} u_2^{\,2}$$

$$\frac{\partial E}{\partial u_1}\frac{du_1}{dt} + \frac{\partial E}{\partial u_2}\frac{du_2}{dt} = -\frac{1}{D}\left[Du_1^2 + Du_2^2\right] \quad (6.16)$$

$$= -\left(u_1^2 + u_2^2\right) \quad (6.17)$$

$$\therefore \frac{\partial E}{\partial u_1}\frac{du_1}{dt} + \frac{\partial E}{\partial u_2}\frac{du_2}{dt} = -\left(u_1^2 + u_2^2\right) \quad (6.18)$$

which is clearly negative definite. So, E (u_1, u_2) is a Liapunov's function for the linear system.

Next we prove that E (u_1, u_2) is also a Liapunov's function for the non-linear system.

Let f_1 and f_2 be two functions of N_1 and N_2 defined by:

$$f_1(N_1, N_2) = a_{11}N_1[-e_1 - N_1 + cN_2] \quad (6.19)$$

$$f_2(N_1, N_2) = a_{22}\left(k_2N_2 - N_2^{\,2} - H_2\right) \quad (6.20)$$

Now we have to show that $\frac{\partial E}{\partial u_1} f_1 + \frac{\partial E}{\partial u_2} f_2$ is negative definite.

By putting $N_1 = \overline{N_1} + u_1$; $N_2 = \overline{N_2} + u_2$ in (6.1) and (6.2), we get:

$$f_1(u_1, u_2) = \frac{du_1}{dt} = a_{11}\,(\overline{N_1} + u_1)\left(-e_1 - \overline{N_1} - u_1 + c\overline{N_2} + cu_2\right)$$

$$= -a_{11}e_1\overline{N_1} - a_{11}\overline{N_1}^2 - a_{11}\overline{N_1}u_1 + ca_{11}\overline{N_1}\,\overline{N_2} + ca_{11}\overline{N_1}u_2 - e_1a_{11}u_1 - a_{11}\overline{N_1}u_1 + ca_{11}\overline{N_2}u_1 + ca_{11}u_1u_2 - a_{11}u_1^2$$

$$= -a_{11}\overline{N_1}u_1 + ca_{11}\overline{N_1}u_2 + u_1a_{11}\left(-e_1 - \overline{N_1} + c\overline{N_2}\right) - a_{11}u_1^2 + ca_{11}u_1u_2$$

$$\Rightarrow f_1(u_1, u_2) = \frac{du_1}{dt} = -a_{11}\overline{N_1}u_1 + ca_{11}\overline{N_1}u_2 + F(u_1, u_2) \quad (6.21)$$

where

$$F(u_1, u_2) = -a_{11}u_1^{\,2} + ca_{11}u_1u_2 \quad (6.22)$$

Also

$$f_2(u_1, u_2) = \frac{du_2}{dt} = a_{22}\left(k_2(\overline{N_2} + u_2) - (\overline{N_2} + u_2)^2 - H_2\right)$$

$$\Rightarrow f_2(u_1, u_2) = \frac{du_2}{dt} = \left(a_{22}k_2 - 2a_{22}\overline{N_2}\right)u_2 + G(u_1, u_2) \quad (6.23)$$

where

$$G(u_1, u_2) = -a_{22}u_2^{\,2} \quad (6.24)$$

From (6.9)

$$\frac{\partial E}{\partial u_1} = Au_1 + Bu_2 \quad (6.25)$$

$$\frac{\partial E}{\partial u_2} = Bu_1 + Cu_2 \quad (6.26)$$

Now

$$\frac{\partial E}{\partial u_1} f_1 + \frac{\partial E}{\partial u_2} f_2 = (Au_1 + Bu_2)\left(-a_{11}\overline{N_1}u_1 + ca_{11}\overline{N_1}u_2 + F(u_1, u_2)\right) + (Bu_1 + Cu_2)\left[\left(a_{22}k_2 - 2a_{22}\overline{N_2}\right)u_2 + G(u_1, u_2)\right] \quad (6.27)$$

$$\frac{\partial E}{\partial u_1} f_1 + \frac{\partial E}{\partial u_2} f_2 = -(u_1^{\,2} + u_2^{\,2}) + (Au_1 + Bu_2)\,F(u_1, u_2) + (Bu_1 + Cu_2)\,G(u_1, u_2) \quad (6.28)$$

Introducing polar co-ordinates $u_1 = r\cos\theta$, $u_2 = r\sin\theta$, the equation (6.28) can be written as:

$$\frac{\partial E}{\partial u_1} f_1 + \frac{\partial E}{\partial u_2} f_2 = -r^2 + r[(A\cos\theta + B\sin\theta)F(u_1, u_2) + (B\cos\theta + C\sin\theta)G(u_1, u_2)] \quad (6.29)$$

Let us denote the largest of the numbers $|A|$, $|B|$ and $|C|$ by K.

Our assumptions imply that $|F(u_1, u_2)| < \frac{r}{6K}$ and $|G(u_1, u_2)| < \frac{r}{6K}$ for all sufficiently small r > 0.

So,

$$\frac{\partial E}{\partial u_1} f_1 + \frac{\partial E}{\partial u_2} f_2 < -r^2 + \frac{4Kr^2}{6K} = -\frac{r^2}{3} < 0 \quad (6.30)$$

Thus E (u_1, u_2) is a positive definite function with the condition that $\frac{\partial E}{\partial u_1} f_1 + \frac{\partial E}{\partial u_2} f_2$ is negative definite.

$\therefore$ The equilibrium state E_5 is "**asymptotically stable**" globally.

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