VOL. 6, NO. 11, NOVEMBER 2011 ISSN 1819-6608
ARPN Journal of Engineering and Applied Sciences

www.arpnjournals.com

NUMERICAL ANALYSIS OF COMBINED HEAT TRANSFER IN THE LAMINAR BOUNDARY LAYER

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ABSTRACT

The effects of thermal radiation upon the laminar flow of an absorbing, emitting, scattering gray fluid over an adiabatic flat plate are investigated. The combined radiation and forced convection in the boundary layer leads to a set of partial and integro-differential equations. In this work, three governing equations are setting as partial differential equations. A finite difference scheme is used to transform the resulting equations into an ordinary differential equation system which is solved numerically. Results for the temperature profiles across the boundary layer and the recovery factor along the flat plate are presented. Comparison of these results with exact solutions shows that the two flux model is simpler and accurate enough to treat the interaction of the thermal radiation with the laminar boundary layer.

Keywords: combined heat transfer, laminar boundary layer, thermal radiation, laminar flow, differential equations.

Nomenclature

A	constant in the two-flux model
Cp	specific heat a constant pressure
E1, E2	integro-exponential functions of first and second class, respectively.
E∞	Eckert number, based in u∞
f	dimensionless stream function
g	gravity
G	irradiation
G*	dimensionless irradiation
I	intensity of radiation
k	thermal conductivity
N	dimensionless conduction-radiation parameter
Nu	Nusselt number
Pr	Prandtl number
q	heat-flux rate
Re	Reynolds Number
Q	dimensionless heat-flux rate
T	absolute temperature
u, v	velocity components in x- and y- directions respectively
Wo	scattering parameter
x	distance along surface
y	distance perpendicular to surface
β	thermal expansion coefficient
δ	boundary-layer thickness
ε	emissivity
η	dimensionless coordinate in the y- direction
θ	dimensionless temperature
ρ	density
σ	Stefan-Boltzmann constant
ξ	dimensionless coordinate in the x- direction
κ	extinction coefficient
κa,	absorption coefficient
κb	dispersion coefficient
τ	optical thickness
ψ	stream function
μ	dynamic viscosity
ν	kinematics viscosity

Subscripts

c	convection
k	conduction
x, y	refers to x- and y-direction
o	refers to To, reference temperature
∞	refers to free stream
w	refers to wall

Superscripts

r	radiation
+	y-direction > 0
-	y-direction < 0

INTRODUCTION

In fluid flow at high velocities over solid surfaces, temperature increases due to frictional effects, commonly known as aerodynamics heating, become important. At the same time, assumption of constant fluid properties may no longer be valid because of steep temperature gradients within the boundary layer. Furthermore, when the temperatures get high enough, fluid molecules in the boundary layer may emit, absorb and scatter thermal radiation. In these cases, both conduction and radiation heat transfer must be included in the energy equation. The treatment of combined conduction and radiation leads to a nonlinear integro-differential energy equation.

Therefore, the exact solution of the governing equations is seldom possible and typically the analyst resorts to approximate formulations, especially considering the optically thin and thick gas limits. Oliver and McFadden [1] and Taitel and Hartnett [2] used iterative approaches to solve the problem of radiative interactions with boundary layer flows of an absorbing and emitting fluid over flat plates. The former investigators were concerned with an isothermal flat plate, while the later ones considered an adiabatic flat plate. Boles and

VOL. 6, NO. 11, NOVEMBER 2011 ISSN 1819-6608
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Ozisik [3] included the effects of scattering of radiation upon compressible boundary layer flow over and adiabatic flat plate, treating the radiation part of the problem exactly with the normal-mode expansion technique. Although these solutions are available in the literature, they are still complex and time-consuming for engineering applications.

The primary purposes of this study is to investigate the combined radiation and forced convection heat transfer of an absorbing, emitting and scattering compressible fluid over an adiabatic plate, using three different radiation models: the thin gas approximation, the thick gas limit and the two flux model. Effects of viscous dissipation and variable properties are also considered. The two-flux model is used according to the procedure indicated by Malpica *et al*., [4] for the case of interaction of radiation with laminar forced convection. The simplification allows converting the equation of radiative transfer into two coupled nonlinear differential equations of second order (Tremante and Malpica [5]) instead of a nonlinear integro-differential equation for the exact formulation (Siegel and Howell [6]). The resulting system of partial differential equations including continuity, momentum, energy and radiation equations is solved using a finite difference scheme (method of columns). Results of the temperature profiles in the boundary layer for different axial positions are obtained for the full range of the boundary layer optical thickness. Results are reported for the recovery factor and scattering effects are also investigated. Comparisons of two-flux results with those obtained using exact formulation reveal excellent agreement.

PROBLEM FORMULATION

This analysis considers the interaction of radiation with forced convection in the laminar boundary layer flow of an absorbing, emitting, isotropically scattering and compressible fluid over a flat plate. It is assumed that the fluid is a perfect gas and is gray, the viscosity varies linearly with temperature, the specific heat and the Prandtl number are constant and the external flow temperature T_∞ is uniform. The surface of the wall is opaque, gray and is a diffuse emitter and reflector and is impervious to heat flow. A sketch of geometry of the physical model and the coordinate system is shown in Figure-1.

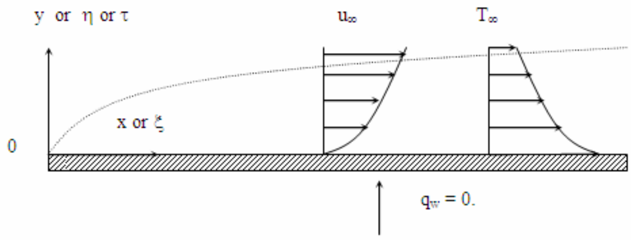

Figure-1. Geometry and coordinate system.

The mathematical model for the assumed physical problem is prescribed by the conservation equation of mass, momentum and energy. These equations are:

Continuity

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \tag{1}$$

Momentum

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \tag{2}$$

Energy

$$\rho C_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = \mu\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) - \frac{\partial q_r}{\partial y} \tag{3}$$

The last term in the right-hand side is the divergence of the radiative heat flux and according to Ozisic [7] is given by:

$$\frac{\partial q_{r,y}}{\partial y} = \kappa_a\Big[\left(4\sigma T^4 - 2\varepsilon_w E_2(\tau)\sigma T_\omega^4\right) - 4(1-\varepsilon_w)E_2(\tau)\int_0^\infty \sigma T^4 E_2(\tau')d\tau' - 2\int_0^\infty \sigma T^4 E_1(|\tau-\tau'|)d\tau'\Big] \tag{4}$$

It means that the energy equation (3) is a nonlinear integro-partial differential equation.
The boundary conditions are:

$$u = v = 0,\ q_w = q_{k,w} + q^r_w = 0 \ \text{at} \ y = 0 \tag{5.a}$$

$$u = u_\infty,\ T = T_\infty \ \text{at} \ y \rightarrow \infty \tag{5.b}$$

The foregoing continuity, momentum and energy equations can be transformed by the application of the standard transformation used in the analysis of the non radiating boundary layer heat transfer.
We define a stream function as:

$$\psi(x,y) \equiv f(\eta)(\nu_o x u_\infty)^{1/2} \tag{6}$$

a new independent variable $\eta(x,y)$ is defined as:

$$\eta \equiv \left(\frac{u_\infty}{\nu_o x}\right)^{1/2}\int_0^y \frac{\rho}{\rho_o}dy' \tag{7}$$

and a dimensionless temperature:

$$\theta = \frac{T}{T_o} \tag{8}$$

Then the continuity equation (1) is identically satisfied. The momentum and energy, equation (2) and (3), respectively are transformed to:
Momentum

$$\frac{d^3f}{d\eta^3} + \frac{1}{2}f\frac{d^2f}{d\eta^2} = 0 \tag{9}$$

The boundary conditions for the momentum equation (9) are:

$f = f' = 0$ at $\eta = 0$ (10.a)
$f' = 1$ at $\eta \rightarrow \infty$ (10.b)

Energy

$$\frac{1}{P_r}\frac{\partial^2\theta}{\partial\eta^2}+\frac{1}{2}f\frac{\partial\theta}{\partial\eta}=\frac{\partial f}{\partial\eta}\xi\frac{\partial\theta}{\partial\xi}+\xi\frac{\partial Q^r}{\partial\tau}-E_\infty\left(\frac{\partial^2 f}{\partial\eta^2}\right)^2 \quad (11)$$

Where various non-dimensional quantities are defined as follows:

$$\xi=\frac{4\sigma T_o^3\kappa_o x}{\rho_o C_p u_\infty}=\frac{(\kappa_o x)^2}{NP_r R_{e,x}}$$ Independent variable along the wall (12)

$$N=\frac{\kappa_o k_o}{4\sigma T_o^3}$$ Conduction-radiation parameter (13)

$$R_{e,x}=\frac{u_\infty x}{\nu_o}$$ Reynolds Number (14)

$$\tau=(\xi NP_r)^{1/2}\eta$$ Optical variable (15)

$$Q^r=\frac{q_r}{4\sigma T_o^4}$$ Dimensionless radiative heat flux (16)

$$E_\infty=\frac{u_\infty^2}{C_p T_o}$$ Eckert number (17)

When the three radiation models under analysis are considered in the energy equation (11), the following expressions results:

a) Thin gas approximation:

$$\frac{1}{\Pr}\frac{\partial^2\theta}{\partial\eta^2}+\frac{1}{2}f\frac{\partial\theta}{\partial\eta}=\frac{\partial f}{\partial\eta}\xi\frac{\partial\theta}{\partial\xi}+\xi\left[\left(\theta^4-1\right)+\frac{\varepsilon_w}{2}\left(1-\theta_w^4\right)\right]-E_\infty\left(\frac{\partial f\theta}{\partial\eta^2}\right) \quad (18)$$

with the following boundary conditions

$$-N\frac{\partial\theta}{\partial\eta}+\frac{\varepsilon_w}{4}(\xi N\Pr)^{\frac{1}{2}}\left[\left(\theta_w^4-1\right)-2(\xi N\Pr)^{\frac{1}{2}}\int_0^\infty\left(\theta^4-1\right)E_2(\tau)d\eta\right]=0$$

at $\eta = 0$ (19.a)

$\theta = 1$ at $\eta \to \infty$ (19.b)

b) Thick gas limit:

$$\frac{1}{\Pr}\left(1+\frac{4\theta^3}{3N}\right)\frac{d^2\theta}{d\eta^2}+\frac{1}{2}f\frac{d\theta}{d\eta}+\frac{4\theta^2}{NP_r}\left(\frac{d\theta}{d\eta}\right)^2=-E_\infty\left(\frac{d^2 f}{d\eta^2}\right) \quad (20)$$

which considers

$$q_y^r=-\frac{4\sigma}{3\kappa}\frac{\partial T^4}{\partial y}$$ Rosseland expression (21)

in this case a similarity solution is obtained.

The boundary conditions are:

$$\left(\frac{d\theta}{d\eta}\right)_w=0$$ at $\eta = 0$ (21.a)

$\theta = 1$ at $\eta \to \infty$ (21.b)

The Two-Flux model is based on the assumption that radiation in the medium may be represented by two uniform fluxes, one in the forward hemisphere and the other in the backward hemisphere (Tremante and Malpica [5]). In principle this formulation replaces the equation of radiative transfer by two-coupled nonlinear partial differential equations, and the expression for the divergence of the radiative flux is given as:

$$\frac{\partial Q^r}{\partial\tau}=(1-W_o)\left(\theta^4-G^*\right) \quad (22)$$

$$G^*=\frac{G}{4\sigma T_o^4} \quad (23)$$

where G^* is the dimensionless total incident irradiation.

according to the two-flux model is coupled with

$$\frac{\partial^2 G^*}{\partial\eta^2}=-A\xi NP_r(1-W_o)(\theta^4-G^*) \quad (24)$$

when equation (22) is substituted in equation (10) the energy equation is given by:

$$\frac{1}{P_r}\frac{\partial^2\theta}{\partial\eta^2}+\frac{1}{2}f\frac{\partial\theta}{\partial\eta}=\frac{df}{d\eta}\xi\frac{\partial\theta}{\partial\xi}+(1-W_o)\xi\left(\theta^4-G^*\right)-E_\infty\left(\frac{d^2 f}{d\eta^2}\right)^2 \quad (25)$$

the boundary conditions are:

$$\left(\frac{\partial\theta}{\partial\eta}\right)_w+\frac{1}{AN}\left(\frac{\partial G^*}{\partial\eta}\right)_w=0$$ at $\eta = 0$ (26.a)

$\theta = 1$ at $\eta \to \infty$ (26.b)

and for the irradiation equation (20)

$$\left(\frac{\partial G^*}{\partial\eta}\right)_w=-A(\xi NP_r)^{1/2}\frac{\varepsilon_w}{2(2-\varepsilon_w)}\left(\theta_w^4-G_w^*\right)$$ at $\eta=0$ (27.a)

$$\left(\frac{\partial G^*}{\partial\eta}\right)_\infty=-\frac{A}{2}(\xi NP_r)^{1/2}\left(1-G_\infty^*\right)$$ at $\eta\to\infty$ (27.b)

With the initial condition

$\theta = \theta_o(\eta)$ at $\xi = 0$ (28)

where $\theta_o(\eta)$ is the solution of the non-radiative case.

The two-flux method is particularized for the model of Milne-Eddington, in that case A = 3. (Malpica *et al*., [4], Tremante and Malpica [5], Siegel and Spuckler [8])

The momentum and energy equations are uncoupled; therefore, the former is previously solved and the functions f (η), f'(η) and f''(η) appearing in the energy equation are determined. Then, the energy equation is solved for the optically thin (18) and thick limits (20). In the case of the thick gas limit a similarity solution is obtained. For the two-flux model the energy equation (25) is solved coupled to the irradiation equation (24) along discrete fluid lines (ξ) perpendicular to the plate. The intervals of the lines in the ξ-direction were chosen on an

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equal logarithmic basis to better reproduce the initial development of the boundary layer. Upwind values were used in the convective terms according to Patankar [9] recommendations. At $\xi = 0$, the solution taken corresponds to the non-radiation problem.

The asymptotic condition at $\eta \to \infty$, was verified by checking an error defined as:

$$\delta_i^2 = \left(f_i''\right)^2 + \left(\theta_i'\right)^2 \tag{29}$$

satisfying the condition where:

$$\delta_i^2 \leq 10^{-6} \tag{30}$$

RESULTS AND DISCUSSIONS

Figure-2 shows non-scattering temperature profiles in the boundary layer over an adiabatic flat plate as function of η at several values of the parameter ξ. The temperature profiles are presented for the three radiation models considered; optically thin gas, two-flux model (A = 3) and the thick gas limit. The results were obtained for $P_r = 0.7$, $E_\infty = 2.3$, $\varepsilon_w = 1.0$, $N = 0.6$ and $W_o = 0$ (non-scattering). These conditions allow viscous energy dissipation and thermal radiation to be significant within the boundary layer and also facilitate comparisons with Taitel and Hartnett [2] results. The temperature profile for $\xi = 0$, non-radiating case, presents the highest wall temperature and a zero temperature gradient at the surface. As the value of ξ increases radiation tends to reduce the temperature in the boundary layer and a positive temperature gradient respect to η establishes to the wall. The magnitude of this positive temperature gradient is controlled by the balance between the radiative heat flux from the wall and the conductive heat flux to the surface. In Table-1 temperature values calculated by the two-flux model are compared with those obtained using the optically thin approximation. These values are also show in Figure-2. Both models predict the same results for $\xi < 4.10^{-4}$. Hence, in this region the boundary layer may be assumed to be optically thin. Table-2 compares two-flux results with those reported by Taitel and Hartnett [2] for the same conditions ($P_r = 0.7$, $E_\infty = 2.3$, $\varepsilon_w = 1.0$, $N = 0.6$). We note from this table, that the two-flux values are in excellent agreement with the exact solution reported by Taitel and Hartnett [2]. In Table-3 we can observed that the two-flux results approach the similarity solution obtained by the optically thick limit approximation, for values of ξ of the order unity or larger. Figure-3 presents the recovery factor, $R = \dfrac{2(\theta_w - 1)}{E_\infty}$ as a function of ξ. Here it may be seen that the flux-model matches in a continuous manner the thin approximation and the thick gas limit.

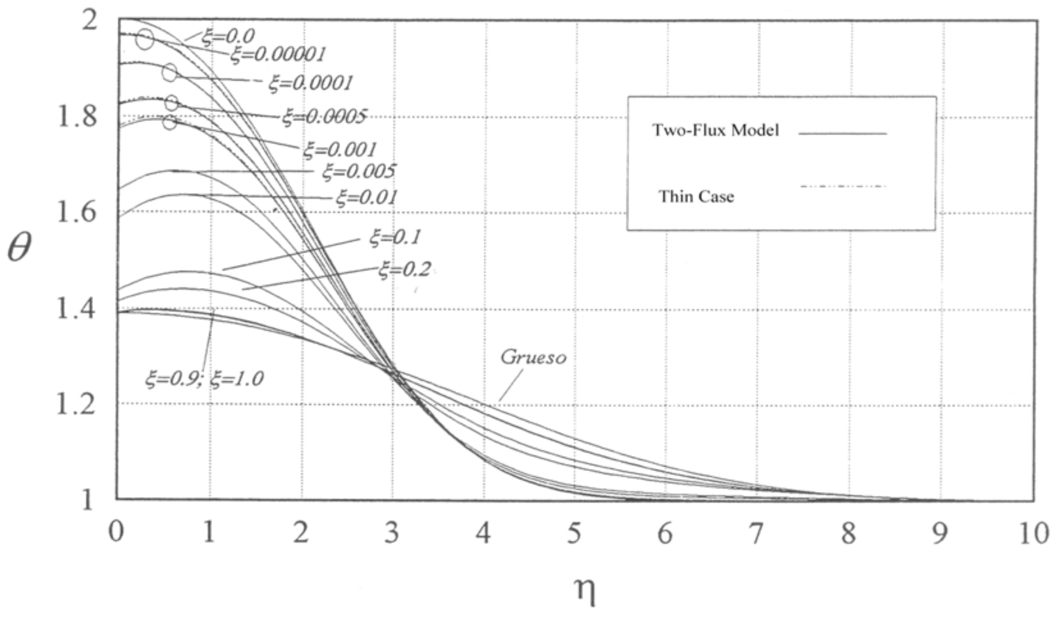

Figure-2. Temperature profiles in forced convection over an adiabatic flat plate. ($P_r = 0.7$, $E_\infty = 2.3$, $\varepsilon_w = 1.0$, $N = 0.6$, $A = 3$, $W_o = 0$).

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Table-1. Dimensionless temperature comparisons of two-flux method with optically thin limit.

	ξ = 0.00001		ξ = 0.0001		ξ = 0.0005		ξ = 0.001	
η	**Two-flux method**	**Thin limit**	**Two-flux method**	**Thin limit**	**Two-flux method**	**Thin limit**	**Two-flux method**	**Thin limit**
0	1.99	1.99	1.92	1.92	1.83	1.83	1.79	1.79
1	1.86	1.86	1.85	1.85	1.77	1.78	1.78	1.79
2	1.58	1.58	1.54	1.54	1.57	1.56	1.53	1.54
3	1.27	1.27	1.29	1.29	1.26	1.26	1.26	1.26

Table-2. Dimensionless temperature comparisons of two-flux method with taitel and Hartnett [2] results.

	ξ = 0.0005		ξ = 0.005		ξ = 0.2	
η	**Two-flux method**	**Ref** [2]	**Two-flux method**	**Ref** [2]	**Two-flux method**	**Ref** [2]
0	1.83	1.83	1.65	1.65	1.43	1.44
1	1.79	1.76	1.67	1.68	1.43	1.45
2	1.55	1.56	1.50	1.50	1.37	1.38
3	1.27	1.28	1.27	1.28	1.25	1.28
4	1.08	1.08	1.17	1.16	1.15	1.15
5	1.03	1.01	1.08	1.09	1.07	-----

Table-3. Dimensionless temperature comparisons of two-flux method with optically thick limit.

η	***ξ = 0.1***	***ξ = 0.2***	***ξ = 1.0***	**Thick limit**
0	1.46	1.44	1.37	1.37
1	1.49	1.46	1.37	1.35
2	1.38	1.36	1.32	1.32
3	1.25	1.25	1.27	1.25
4	1.14	1.13	1.18	1.21
5	1.09	1.08	1.12	1.11

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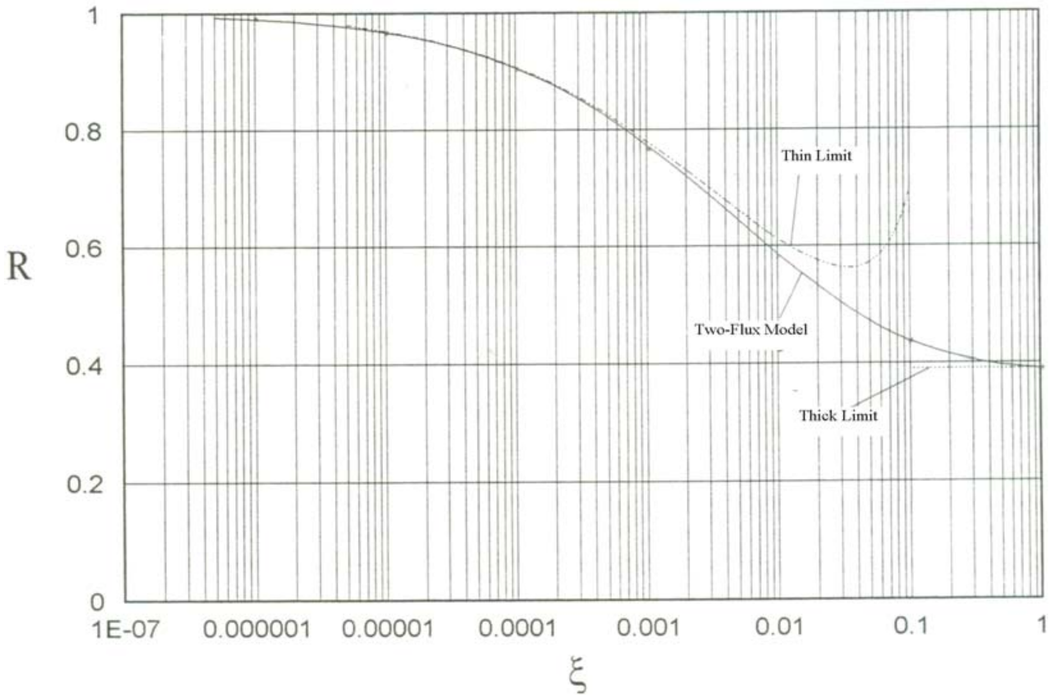

Figure-3. Recovery factor in forced convection over an adiabatic flat plate. (P_r = 0.7, E_∞ = 2.3, ε_w = 1, N = 0.6, A = 3, W_o = 0).

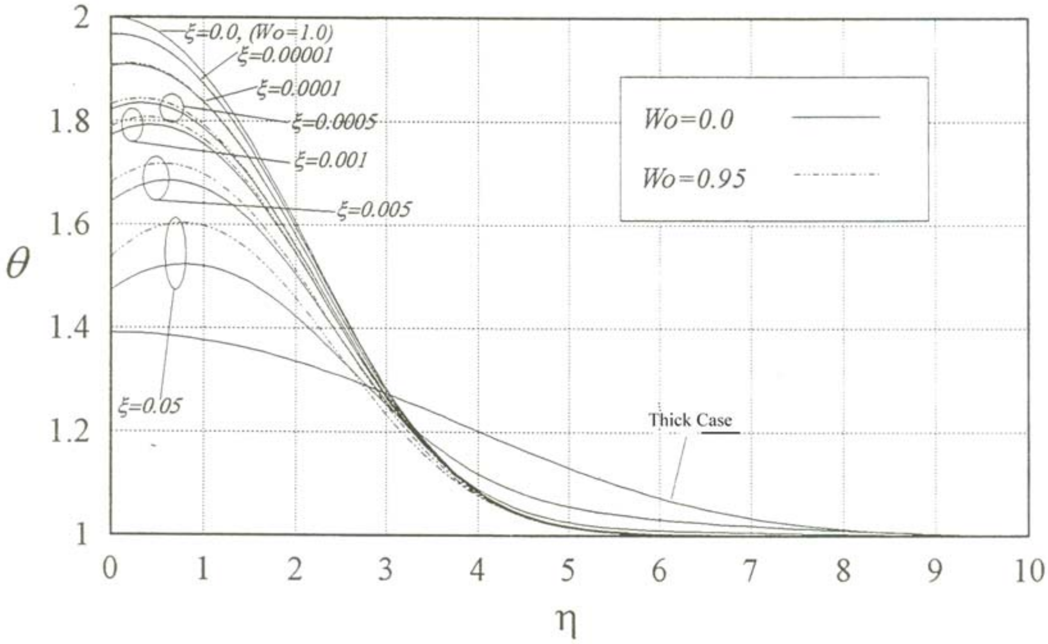

Figure-4. Temperature profiles in forced convection over an adiabatic flat plate effect of scattering. W_o. (P_r = 0.7, E_∞ = 2.3, ε_w = 1.0, N = 0.6, A = 3

REFERENCES

[1] Oliver C.C. and McFadden P.W. 1999. The Interaction of Radiation and Convection in the Laminar Boundary Layer. Journal of Heat Transfer. 88C: 205-213.

[2] Taitel Y. and Hartnett J.P. 1993. Equilibrium Temperatures in a Boundary-Layer Flow over a Flat Plate of Absorbing-Emitting Gas. Paper 66-WA/HT-48.

[3] Boles M.A. and Ozisik M.N. 1995. The Effect of Scattering of Radiation upon Compressible Boundary

Layer over an Adiabatic Flat Plate. Journal of Heat Transfer. pp. 311-313.

[4] Malpica F., Campo A. and Tremante A. 1996. Contribution of Thermal Radiation to the Temperature Profile of Semitransparent Materials. High Temperature-High Pressures. 18: 35-41.

[5] Tremante A. and Malpica F. 1994. Contribution of Thermal Radiation to the Temperature Profile of Ceramics Composite Materials. Journal of Engineering for Gas Turbines and Power. 116: 583-586.

[6] Siegel R. and Howell J.R. 2002. Thermal Radiation Heat Transfer. 2nd Ed., Hemisphere, Washington D.C. USA. p. 501.

[7] Ozisik K.M. 1993. Radiative Transfer and Interaction with Conduction and Convection. John Wiley and Sons, New York, USA. pp. 521-522.

[8] Siegel R. and Spuckler C.M. 1994. Approximate Solution Methods for Spectral Radiative Transfer in High Refractive Index Layers. Int. J. Heat Mass Transfer. 17(Suppl. 1): 403-413.

[9] Patankar S.V. 1982. Numerical Heat Transfer and Fluid Flow. Hemisphere, Washington D.C. USA. pp. 83-84.

[10] Pereira V. 2001. PASVA3: An adaptive finite differences FORTRAN program for first order nonlinear ordinary differential equation problems. pp. 67-68.