ANALYTICAL INVESTIGATION OF Prestressed Concrete Structures Incorporating Combined Post-Tensioned and Post-Compressed Reinforcements

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ABSTRACT
Although strengthening of structures by post-tensioning is much more common in practice, the authors did locate innovative literature on the theory and applications of post-compression which form the basis for this study. This work analytically assesses the behaviour and criteria for optimal performance of prestressed concrete structures under the combined actions of post-tensioning and post-compression. Magnel type diagrams are also developed to show the feasible solution regions for different loading conditions. The findings revealed that the prestressing forces in the post-tensioned tendons are independent of the eccentricity ratio, but increases perfectly curvilinearly up to prestressing ratio of 0.7 beyond which the technique becomes impracticable. On this basis, there is a realistic reduction of 50 percent in the eccentricity and possibly the overall depth of the concrete section over the conventional technique. This qualifies this structural innovation as a reliable candidate for long span structures.

Keywords: prestressed concrete, post-compression, post-tensioned, Magnel type diagram.

INTRODUCTION
Prestressed concrete structures are an attractive alternative for long-span bridges, and have been used worldwide since the 1950s. Savings in materials and maintenance and life-cycle costs are additional advantages. The extensive application of prestressed concrete (PC) to buildings, military constructions and civil infrastructure has been generally enhanced by awareness with its principles, practice and further advances in its design and construction [1, 2]. Sriskandan [3] and Smyth [4] provided excellent reviews on research advances, developments and achievements of PC bridges in the United Kingdom in comparison with practices in the United States and Europe. The performance of PC highway bridges in the USA was also investigated by Dunker and Rabbat [5]. Recent bridge inventories in the Western world and some Asian countries have shown that such structures constitute more than 50 percent of existing bridges, and are rapidly replacing existing structural steel bridges. Furthermore, the use of external prestressing techniques has been growing rapidly since its evolution in the early 1950s and many investigations have been reported on externally PC structures [6]. Indeed the amount of literature related to advances in PC structures is quite extensive.

One important technique of prestressing that has not received wide practical application is the use of slender compressed steel bars in concrete structures which was first proposed in the early 1950s by Kurt Billig [7]. Post-compressing a structure is similar to post-tensioning a structure except that the member is subjected to axial tension rather than axial compression. Neither of the two separate patents filed by Billig in West Germany and Austria in 1953 and 1956 respectively was put to immediate application until the design methodology and construction details for the use of a post-compression system in concrete structures were fully developed in the early 1970s by Reiffenstuhl [8, 7]. He developed a method of applying post-compression in concrete structure to cancel the axial effects associated with post-tensioning. By simultaneously post-tensioning and post-compressing a member to the same extent, the tension and compression axial forces cancel out. Therefore, this technique guarantees the minimum section possible to satisfy the serviceability and ultimate conditions in PC structures.

This paper presents the analytical investigation of PC structures incorporating both post-tensioned tendons and post-compressed bars. Mathematical expressions are derived for the analysis of such prestressed concrete structural beams for transfer and service conditions. Magnel type diagrams are also developed to show the feasible regions. Finally, an analytical example of a typical prestressed concrete beam is given and the structural behaviour is properly assessed.

REVIEW OF LITERATURE
Advances of post-compression systems in conventional prestressed concrete

Post-compressing a structure using slender bars does have associated problems. The problems associated with achieving compression in a slender bar throughout the member, without significant friction losses or buckling of the bar in the duct as well as providing tension anchorages that will carry design loads during the life of the structure have been effectively solved by Reiffenstuhl [8, 9]. The technique, known for better economic advantage, has found application in the construction of new and the strengthening of existing PC structures.

An unusually slender 76-metre single span Alm Bridge in Austria having a depth of about 2.50 metres corresponding to span/depth ratio of 30.4 was the first
prestressed concrete bridge with post-compressed reinforcements [8]. Compressed steel rods in specially developed ducts arched to mirror the conventional post-tensioned ones were utilized to induce tension in the concrete to counteract compressive stresses that would otherwise occur at the extreme top fibre. This ensured that all dead loads were carried absolutely by the prestress while the section carried live load only. Joints between duct sections were made by simply butting the bars within stiff, square cross-section lengths of duct that accurately fit the bars. Anchorage assembly and jacking method were also developed by Reiffenstuhl [8, 9].

In addition, strengthening of a 7-year old, 36.3-metre span PC folded-plate roof structure of an athletic building was successfully achieved with the post-compression method [10]. Analysis of the existing V-members indicated that almost all of the compression capacity of the concrete had been utilized in the original design. Although the existing steel tendons could have provided adequate force, pretensioning them would have overstressed the anchorages. Therefore, the post-compression technique was applied to the exterior of each plate of all V-members to preserve the architectural character of the roof and also due to the necessity of using the athletic building during strengthening.

Reiffenstuhl [11] showed ingenuity in the design of the pedestrian bridge over the Rupert-Mayer-Straße in Munich using a prestressed concrete structure with pre-compressed reinforcement. Initially the bridge was conceived in U-section over three spans in order to minimize the depth of the cross-section. The bridge was thereafter changed to a single span, when it was realized that the intermediate piers, made of circular columns with bulky cross section, would affect the traffic underneath as well as the overall architectural appearance. Post-compressing the reinforcements in the web increased the bending capacity of the cross-section such that the bridge could be executed as one span girder. The savings achieved by omitting the piers was much higher than the costs for additional reinforcements.

Similar to the conventional prestressed tendon profile, Klaiber [12], Reiffenstuhl [9-11] and more importantly Zheng et al., [13] have shown very clearly that post-compressed bar profile can be straight, draped or harped. Furthermore, more recent advances in the design of post-compression systems such as the ducts, anchorage assembly and jacking methods developed in Japan and China to check the buckling of slender steel bars in prestressed concrete structures are presented by Zheng et al., [13].

MATERIALS AND METHODS

Elastic flexural analysis in service

The design of a prestressed concrete structure involves many considerations, the most important of which is the determination of stress distributions in the individual members of the structure. The level of prestress and the layout of the tendons in a member are usually determined from the serviceability requirements of that member. For the serviceability requirements to be satisfied, a reasonably accurate estimate of the magnitude of prestress is needed. This requires reliable procedures for the determinations of both the instantaneous and the time-dependent losses of prestress that is at transfer and service conditions, which are the two critical stages in prestressed concrete design for serviceability. The serviceability analysis of prestressed concrete beams reinforced with combined post-tensioned and post-compressed tendons is carried out to study the elastic flexural behaviour.

General assumptions and sign conventions

The analysis of stresses developed in the prestressed concrete structures under gravity loads and combined action post-tensioned tendon and post-compression reinforcing system is based on the assumptions that strain distribution across the depth of the member is linear, concrete is a homogeneous elastic material, and both concrete and steel behave elastically within the range of working stresses notwithstanding the small amount of creep which occurs in both materials under sustained loading. The sign conventions of positive sagging moments and negative hogging moments, and positive compressive forces and negative tensile forces are applied in this analysis. In addition, eccentricity of post-tensioned tendons below the concrete centroid and eccentricity of post-compressed bars above the concrete centroid are positive.

Flexural analysis of prestressed concrete in serviceability limit state

The three most common serviceability limit states are the checks on the prestressing tensile steel and compressive concrete stresses, crack control and deformation. Span/depth ratios and the level of prestress are governing parameters for deformation control. Most codes of practice or design standards group prestressed concrete members in different classes which are usually based on the allowable level of flexural stress for effective crack control. However, limitation on stresses is the fundamental factor that defines the serviceability of a prestressed concrete structural member. Limits are imposed on the concrete stresses to avoid longitudinal cracking and restrict creep deformation. Also, limits are set for the effective prestress in tendons after allowance for losses to prevent stresses in the prestressing steel under serviceability condition which could lead to inelastic deformation of tendon.

Flexural stresses are the result of bending moments due to gravity or imposed loads and eccentric prestressing forces. Major codes of practice or design standards such as BS 8110 [14], ACI 318-08 [15] and AS 3600 [16] suggest that expected load rather than characteristic load should be used in design in order to produce a best estimate for serviceability requirements. A simply supported prestressed concrete beam of span, L, and preliminary geometry as shown in Figure-1 is reinforced with post-tensioned and post-compressed
reinforcing steels at eccentricities, \( e_B \) and \( e_T \), respectively. The applied gravity loads on the structure are the self weight, \( w_G \), stationary superimposed dead load, \( w_D \), and the superimposed live load, \( w_L \). Only the self weight, \( w_G \), acts at transfer while the full intensity of all the three loads known as service load, \( w_S \), occurs during service after the time-dependent losses in prestress have taken place. The beam has cross-sectional area, \( A \), and bottom and top elastic section moduli \( B_Z \) and \( T_Z \), respectively, for the concrete section. In Figure-1, the symbols \( cgst \) and \( cgsb \) are the centroids of the post-compressed and post-tensioned steels, respectively; \( cg_c \) is the centroid of the gross area of concrete section; \( e_T \) and \( e_B \) are the eccentricities of the post-compressed and post-tensioned steels, respectively.

![Figure-1. Prestressed concrete beam reinforced with straight post-tensioned and post-compressed tendons.](image)

Using the uncracked unsymmetrical section, a summary of the equations of stresses for the transfer and service stages due to the initial prestressing forces \( P_T \) and \( P_B \) before losses in the compressed and tensioned bars respectively can be readily outlined. The stresses at the extreme top and bottom fibres at transfer are

\[
\begin{align*}
\sigma_{B,i} &= \frac{P_T e_T}{A} + \frac{P_B e_B}{T_Z} + \frac{M_i}{Z_T} \leq f_{i,j} \\
\sigma_{T,i} &= \frac{P_T e_T}{T_Z} + \frac{P_B e_B}{A} + \frac{M_i}{Z_T} \geq f_{i,j}
\end{align*}
\]

(1)

Assume residual prestress ratios \( \mu \) and \( \alpha \) after time dependent losses have taken place in relation to the prestressing forces \( P_T \) and \( P_B \), respectively. The stresses in the extreme top and bottom fibres of the concrete section are given by:

\[
\begin{align*}
\sigma_{B,s} &= \mu \left( \frac{P_T e_T}{A} + \frac{P_B e_B}{T_Z} \right) + \alpha \left( \frac{P_B e_B}{A} + \frac{P_B e_B}{Z_B} \right) - \frac{M_s}{Z_T} \leq f_{c,s} \\
\sigma_{T,s} &= \mu \left( \frac{P_T e_T}{T_Z} + \frac{P_B e_B}{A} \right) + \alpha \left( \frac{P_T e_T}{A} + \frac{P_B e_B}{Z_B} \right) - \frac{M_s}{Z_B} \geq f_{c,s}
\end{align*}
\]

(2)

where \( M_i \) and \( M_S \) are the maximum bending moments at transfer and service, respectively and \( M_S \) is given by:

\[
M_S = M_t + M_D + M_L
\]

(5)

Combination of equations (1) and (3) yields:

\[
\frac{M_S}{Z_T} \geq f_{c,s} - f_{c,i} - \left( 1 - \mu \right) \left( \frac{1}{A} + \frac{e_T}{Z_T} \right) P_T - \left( 1 - \alpha \right) \left( \frac{1}{A} + \frac{e_B}{Z_T} \right) P_B \tag{6}
\]

A similar operation on equations (2) and (4) yields:

\[
\frac{M_S}{Z_B} \geq \frac{M_s}{f_{c,i} - f_{c,s}} \tag{7}
\]

where \( M_2 = M_S - M_i \)

\( f_{c,i} \), \( f_{c,s} \) = bottom and top fibre stress, respectively at transfer stage

\( f_{c,i} \), \( f_{c,s} \) = bottom and top fibre stress, respectively at service stage

\( f_{t,i} \), \( f_{t,s} \) = allowable compressive stress at transfer and service stage, respectively

\( f_{t,i} \), \( f_{t,s} \) = allowable flexural tensile stress at transfer and service stage, respectively

**Preliminary section sizing**

From several inequalities that have to be satisfied at any cross-section, it is possible to separate out the design of the cross-section from the design of prestress. By considering pairs of stress limits on the same fibre, but for different load cases, the effects of prestress can be eliminated. The preliminary sizing is more easily achieved by introducing some simplifying assumptions. Assuming little or negligible loss of prestress such that the residual prestress ratios, \( \mu \) and \( \alpha \), are approximately equal to unity, the top and bottom elastic section moduli can be obtained from Equations (6) and (7), respectively as follows:

\[
\frac{Z_T}{f_{c,s} - f_{c,i}} \geq M_S \tag{8a}
\]

\[
\frac{Z_B}{f_{c,i} - f_{c,s}} \geq M_S \tag{8b}
\]

Also, the following general assumptions may be considered for a more reasonable approximation:
\[ \frac{P_b}{A} = 0.5f_{c,s} \text{ or } 0.5f_{c,i} \]  (9a)

\[ \frac{e_B A}{Z_B} \text{ or } Z_T = 2 \]  (9b)

If \( P_t/P_B = \lambda \) and \( e_T/e_B = \beta \), and Equations (9a) and (9b) are applied to Equations (6) and (7), the top and bottom elastic section moduli can be obtained as

\[ Z_T \geq \frac{M_2}{0.5f_{c,s}[1 + \alpha - \lambda(1 - \mu)(1 + 2\beta)] - f_{c,i}} \]  (10a)

\[ Z_B \geq \frac{M_2}{0.5f_{c,i}[3\alpha - 1 + \lambda(1 - \mu)(1 - 2\beta)] - f_{c,s}} \]  (10b)

The inequalities in Equations (8a) and (8b) or Equations (10a) and (10b) can be utilized to determine the minimum size of the cross-section.

**Determination of the feasible solution regions**

Once a suitable cross-section has been found, the prestress can be designed by combining the stress limits in the extreme fibres at transfer and service represented by the four inequalities given by Equations (1) to (4). The feasible tendon zones of the prestressed concrete member can be determined using a construction similar to that of Magnel [17]. The choice of a suitable Magnel type diagram depends on whether \( f_{c,s} < M_s/Z_T \) in Equation (3). By plotting the four inequalities on a graph of \( e_B \) (vertical axis) against \( \frac{A}{P_B} \) (horizontal axis), a series of bound lines are formed here referred to as the Magnel type diagram. Provided the trial section is suitable, these bound lines will always result in a zone showing all feasible combinations of \( P_b \) and \( e_B \) for suitable values of \( \lambda \) and \( \beta \). Plotting the eccentricity on the vertical axis for direct comparison with the cross-section, the Magnel type diagrams for this special case of prestressed concrete beam are shown in Figures 2 and 3.

**RESULTS AND DISCUSSIONS**

### Analytical example

A simply supported prestressed concrete (PC) beam of rectangular section 450 mm \( \times \) 675 mm having a span of 18 metres as shown in Figure-4 is considered for serviceability assessment. It is subjected to a uniform intensity of stationary superimposed dead load and superimposed live load of 10.20kN/m and 5.85kN/m, respectively. A value of 0.85 is assumed for the residual prestress ratios \( \mu \) and \( \alpha \) for the post-tensioned tendons and post-compressed bars respectively, to account for the loss.

![Figure-2. Magnel type diagram for \( f_{c,s} \geq \frac{M_s}{Z_T} \).](image)

![Figure-3. Magnel type diagram for \( f_{c,s} < \frac{M_s}{Z_T} \).](image)
of prestress of about 15 percent of the total jacking force after all the time-dependent losses have taken place. The characteristic concrete strengths at transfer, \( f_{ci} \) and service, \( f_{cu} \) are 40N/mm\(^2\) and 50N/mm\(^2\), respectively. The beam is designed as Class 1 member in accordance with BS 8110-1-1997 [14]. The allowable concrete compressive and tensile stress limits are as specified in Sections 4.3.4 and 4.3.5 of BS 8110-1-1997 [14] for service and transfer conditions, respectively. The unit weight of concrete is 24kN/m\(^3\).

**Figure-4.** Elevation and cross-section of the analytical PC rectangular beam.

Allowable compressive stresses in concrete:
- At transfer, \( f_{c,i} = 0.5f_{ci} = 20N/mm^2 \)
- In service, \( f_{c,s} = 0.33f_{cu} = 16.5N/mm^2 \)

Allowable flexural tensile stresses in concrete:
- At transfer, \( f_{t,i} = -1.0N/mm^2 \) (minus sign indicates tension)
- In service, \( f_{t,s} = 0 \) (no tensile stress is allowed in Class 1 member)

Since \( f_{c,s} < M_{Ts}/Z_T = 27.66 N/mm^2 \), Magnel type diagram shown in Figure-3 satisfies the requirement for the analysis.

Self weight of beam, \( w_G = 7.29 kN/m \)

The total service load \( w_s = w_G + w_D + w_L = 7.29 + 10.20 + 5.85 = 23.34 kN/m \).

Initial moment due to self-weight, \( M_i = w_GL^2/8 = 295.25 kNm \)

Total moment in service, \( M_S = w_SL^2/8 = 945.27 kNm \).

**Figure-5.** Variation of prestressing force in the post-tensioned tendon.

It is, therefore, obvious that \( \lambda < 1 \) for feasible design. Moreover, it can be inferred from Figure-5 that the relationship between the prestressing forces in the post-tensioned and post-compressed steels within the range \( 0.7 < \lambda < 1 \) is cubic polynomial. The mathematical expressions relating the prestressing forces \( P_B \) and \( P_T \) for the maximum and minimum permissible prestressing force conditions are as given by Equations (12a) and (12b) respectively as follows:

\[
\begin{align*}
P_B^3 - 3.7425P_B^2 + 1.9662P_BP_T - 18.052P_T^2 &= 0 \quad \text{(12a)} \\
P_B^3 - 2.534P_B^2 + 1.3313P_BP_T - 12.223P_T^2 &= 0 \quad \text{(12b)}
\end{align*}
\]
Equation (12b) with minimum prestress is preferred for economic design where the design is limited by the permissible tensile stresses. The variation of the maximum difference in the prestressing force in the post-tensioned tendon with the prestressing force ratio is also perfectly quadratic within the range $0 \leq \lambda \leq 0.7$ as shown in Figure-6.

Figure-6. Variation of maximum change in prestressing force.

**Variations in eccentricity for analytical PC beam**

In Figure-7, the minimum and maximum feasible eccentricities of the post-tensioned steel, $e_B$, are plotted against the ratio of the prestressing force in the post-tensioned steel to that of post-compressed steel, $\lambda$, for different ratios of the eccentricity of post-tensioned tendon to that of post-compressed bars, $\beta$. These plots are generated for the vertices A, B, C and D bounding the feasible solution regions of the Magnel type diagram for $f_{cs} < M_b / Z_T$ previously presented in Figure-3. The maximum and minimum eccentricities and prestressing forces are then extracted from the analytical solutions. It is evident that eccentricities are directly affected by prestressing force ratio and eccentricity ratio.

The minimum and maximum post-tensioned tendon eccentricities are greatest for the conventional technique where no post-compressed steel is introduced. On the contrary, the eccentricity, and essentially, the overall depth of concrete section, evidently reduces with the introduction of post-compression. Moreover, the eccentricities further reduce as the percentage of post-compression force increases. Interestingly, it is noteworthy that there is a remarkable reduction of about 50 percent in the eccentricity $e_B$ when the prestressing force ratio, $\lambda$, has a value of 0.7, which indicates a realistic reduction in the overall depth of the cross-section over the conventional prestressing technique without post-compression. Furthermore, much longer concrete structures can be realized with post-compression over the conventional prestressed concrete approach without any considerable change in weight. This concept of design is technically impracticable when the prestressing force ratio approaches unity, because eccentricities approximate to zero while the axial forces cancel out. In addition, the eccentricities decrease for any prestressing force ratio as the eccentricity ratio increase. The variation of eccentricity with prestressing force is perfectly linear when $\beta$ tends towards zero, and cubic or higher polynomial as $\beta$ increases beyond 1.0. The permissible tendon zone, which is the difference between the maximum and minimum tendon eccentricities, is an important aspect in the analysis and design of prestressed concrete structures which determines the duct size required. It can be understood from Figure-8 that the permissible tendon zone variation with prestressing force ratio is linear as $\beta$ approaches zero and higher polynomial when $\beta$ increases beyond 1.0. It is also noteworthy that the permissible tendon zone diminishes as prestressing force and eccentricity ratios increase. Thus, it can be concluded that prestressing force ratios in the range
0.8 ≤ λ ≤ 1 may not be feasible in the practical sense because the cable zone tends to close up.

Figure-8. Permissible post-tensioned tendon cable zones.

CONCLUSIONS

A general overview of the historical background, fundamental principles and previous applications of post-compression techniques in prestressed concrete structures has been covered in this paper. Basic equations and inequalities are also derived, and Magnel type diagrams constructed for general analysis and design of prestressed concrete structures with both post-tensioned tendons and post-compressed bars. In the analytical study of a simple beam, the prestressing forces in the post-tensioned tendons are found to be independent of the eccentricity ratio, but increase quadratically up to a prestressing ratio of 0.7 after which they become asymptotic as the post-compression force equals the post-tension force. On the other hand, the prestressing force ratio and the eccentricity ratio have direct effects on the eccentricities. There is reduction in eccentricities as the post-compression force and eccentricity of post-compressed steel bars increase. This reduction is perfectly linear for concentric post-compression and quadratic or higher polynomial as the eccentricity of the post-compressed steel bars increases. This concept of design is technically impracticable when the prestressing force ratio approaches unity, because eccentricities approximate to zero while the axial forces cancel out.

REFERENCES


