VOL. 7, NO. 8, AUGUST 2012 ISSN 1819-6608
ARPN Journal of Engineering and Applied Sciences

www.arpnjournals.com

A PREY-PREDATOR MODEL WITH AN ALTERNATIVE FOOD FOR THE PREDATOR AND OPTIMAL HARVESTING OF THE PREDATOR

K. Madhusudhan Reddy[1] and K. Lakshmi Narayan[2]
[1]Vardhaman College of Engineering, Hyderabad, India
[2]SLC'S Institute of Engineering and Technology, Hyderabad, Piglipur, Hyderabad, India
E-Mail: kmsreddy.vce@gmail.com

ABSTRACT

The present paper deals with a prey - predator model comprising an alternative food for the predator in addition to the prey, and the predator is harvested under optimal conditions. The model is characterized by a pair of first order non-linear ordinary differential equations. All the four equilibrium points of the model are identified and the criteria for the stability are discussed. The possibility of existence of bioeconomic equilibrium is discussed. The optimal harvest policy is studied with the help of Pontryagin's maximum principle [1]. Finally, some numerical examples are discussed.

Keywords: prey; predator, equilibrium points, stability, bionomic equilibrium, optimal harvesting, normal steady state.

1. INTRODUCTION

Ecology relates to the study of living beings in relation to their living styles. Research in the area of theoretical ecology was initiated by Lotka [2] and by Volterra [3]. Since then many mathematicians and ecologists contributed to the growth of this area of knowledge as reported in the treatises of Paul Colinvaux [4], Freedman [5], Kapur [6, 7] etc. Harvesting of multispecies fisheries is an important area of study in fishery modeling. The issues and techniques related to this field of study and the problem of combined harvesting of two ecologically independent populations obeying the logistic law of growth are discussed in detail by Clark [8, 9]. Chaudhuri [10, 11] formulated an optimal control problem for the combined harvesting of two competing species. Models on the combined harvesting of a two-species prey-predator fishery have been discussed by Chaudhuri and Saha Ray [12]. Biological and bionomic equilibria of a multispecies fishery model with optimal harvesting policy is discussed in detail by Kar and Chaudhari [13]. Recently Archana Reddy [14] discussed the stability analysis of two interacting species with harvesting of both species. Lakshmi Narayan, Pattabhiramacharyulu [15, 16] and Shiva Reddy [17] *et al*., have discussed different prey-predator models in detail. Most of the mathematical models on the harvesting of a multispecies fishery have so far assumed that the species are affected by harvesting only. A population model proposed by Kar and Chaudhuri, (c.f. Harvesting in a two-prey one-predator fishery: Bioeconomic model, ANZIAM J.45 (2004), 443-456) motivated the present investigation.

2. MATHEMATICAL MODEL

The model equations for a two species prey-predator system are given by the following system of non-linear ordinary differential equations employing the following notation:

N_1 and N_2 are the populations of the prey and predator with natural growth rates a_1 and a_2 respectively, α_{11} is rate of decrease of the prey due to insufficient food, α_{12} is rate of decrease of the prey due to inhibition by the predator, α_{21} is rate of increase of the predator due to successful attacks on the prey, α_{22} is rate of decrease of the predator due to insufficient food other than the prey; q_2 is the catch ability co-efficient of the predator, E is the harvesting effort and q_2EN_2 is the catch-rate function based on the CPUE (catch-per-unit-effort) hypothesis. Further both the variables N_1 and N_2 are non-negative and the model parameters $a_1, a_2, \alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}, q_2, E$ and $(a_2 - q_2E)$ are assumed to be non-negative constants.

$$\frac{dN_1}{dt} = a_1N_1 - \alpha_{11}N_1^2 - \alpha_{12}N_1N_2 \tag{1}$$

$$\frac{dN_2}{dt} = (a_2 - q_2E)N_2 - \alpha_{22}N_2^2 + \alpha_{21}N_1N_2 \tag{2}$$

3. EQUILIBRIUM STATES

The system under investigation has four equilibrium states defined by:

a) The fully washed out state with the equilibrium point $\overline{N}_1 = 0; \overline{N}_2 = 0$ (3)

b) The state in which, only the predator survives given by $\overline{N}_1 = 0; \overline{N}_2 = \frac{(a_2 - q_2E)}{\alpha_{22}}$ (4)

c) The state in which, only the prey survives given by $\overline{N}_1 = \frac{a_1}{\alpha_{11}}; \overline{N}_2 = 0$ (5)

d) The co-existent state (normal steady state) given by:

VOL. 7, NO. 8, AUGUST 2012 ISSN 1819-6608
ARPN Journal of Engineering and Applied Sciences

www.arpnjournals.com

$$\overline{N}_1 = \frac{a_1\alpha_{22} - (a_2 - q_2E)\alpha_{12}}{\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}};\ \overline{N}_2 = \frac{(a_2 - q_2E)\alpha_{11} + a_1\alpha_{21}}{\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}} \quad (6)$$

This state would exit only when $a_1\alpha_{22} > (a_2 - q_2E)\alpha_{12}$ (7)

4. STABILITY OF THE EQUILIBRIUM STATES

To investigate the stability of the equilibrium states we consider small perturbations u_1, u_2 in N_1 and N_2 over $\overline{N}_1$ and $\overline{N}_2$ respectively, so that

$N_1 = \overline{N}_1 + u_1$; $N_2 = \overline{N}_2 + u_2$ (8)

By substituting (8) in (1) and (2) and neglecting second and higher powers of the perturbations u_1, u_2, we get the equations of the perturbed state:

$$\frac{dU}{dt} = AU \quad (9)$$

where

$$A = \begin{bmatrix} a_1 - 2\alpha_{11}\overline{N}_1 - \alpha_{12}\overline{N}_2 & -\alpha_{12}\overline{N}_1 \\ \alpha_{21}\overline{N}_2 & (a_2 - q_2E) + \alpha_{21}\overline{N}_1 - 2\alpha_{22}\overline{N}_2 \end{bmatrix} \quad (10)$$

The characteristic equation for the system is:

$$det[A - \lambda I] = 0 \quad (11)$$

The equilibrium state is stable only when the roots of the equation (11) are negative in case they are real or have negative real parts in case they are complex.

The equilibrium states I, II, and III are found to unstable, so we restricted our study to the normal steady state only.

4.1 Stability of the normal steady state

In this case the characteristic equation is

$$\lambda^2 + (\alpha_{11}\overline{N}_1 + \alpha_{22}\overline{N}_2)\lambda + [\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}]\ \overline{N}_1\overline{N}_2 = 0 \quad (12)$$

Since the sum of the roots of (12) is negative and the product of the roots is positive, the roots of which can be noted to be negative. Hence the co-existent equilibrium state is stable.

The solutions curves are:

$$u_1 = \left[\frac{u_{10}(\lambda_1 + \alpha_{22}\overline{N}_2) - u_{20}\alpha_{12}\overline{N}_1}{\lambda_1 - \lambda_2}\right] e^{\lambda_1 t} + \left[\frac{u_{10}(\lambda_2 + \alpha_{22}\overline{N}_2) - u_{20}\alpha_{12}\overline{N}_1}{\lambda_2 - \lambda_1}\right] e^{\lambda_2 t} \quad (13)$$

$$u_2 = \left[\frac{u_{20}(\lambda_1 + \alpha_{11}\overline{N}_1) + u_{10}\alpha_{21}\overline{N}_2}{\lambda_1 - \lambda_2}\right] e^{\lambda_1 t} + \left[\frac{u_{20}(\lambda_2 + \alpha_{11}\overline{N}_1) + u_{10}\alpha_{21}\overline{N}_2}{\lambda_1 - \lambda_2}\right] e^{\lambda_2 t} \quad (14)$$

Where

λ_1, λ_2, are the roots of the equation (12).

5. GLOBAL STABILITY

Theorem: The Equilibrium state $\left(\overline{N}_1, \overline{N}_2\right)$ is globally asymptotically stable.

Let us consider the following Liapunov's function

$$V(N_1, N_2) = N_1 - \overline{N}_1 - \overline{N}_1 \ln\left[\frac{N_1}{\overline{N}_1}\right] + l\left\{N_2 - \overline{N}_2 - \overline{N}_2 \ln\left[\frac{N_2}{\overline{N}_2}\right]\right\} \quad (15)$$

where 'l' is positive constant ,to be chosen later

Differentiating V w.r.t 't', we get

$$\frac{dV}{dt} = \left(\frac{N_1 - \overline{N}_1}{N_1}\right)\frac{dN_1}{dt} + l\left(\frac{N_2 - \overline{N}_2}{N_2}\right)\frac{dN_2}{dt} \quad (16)$$

Substituting (1) and (2) in (16), we get

$$\frac{dV}{dt} = \left\{-\alpha_{11}\left(N_1 - \overline{N}_1\right)^2 - \alpha_{12}\left(N_1 - \overline{N}_1\right)\left(N_2 - \overline{N}_2\right)\right\} + l\left\{\alpha_{21}\left(N_1 - \overline{N}_1\right)\left(N_2 - \overline{N}_2\right) - \alpha_{22}\left(N_2 - \overline{N}_2\right)^2\right\}$$

Choosing $l = \frac{\alpha_{12}}{\alpha_{21}}$ and with some algebraic manipulation yields

$$\frac{dV}{dt} = -\alpha_{11}(N_1 - \overline{N}_1)^2 - \frac{\alpha_{12}}{\alpha_{21}}\alpha_{22}(N_2 - \overline{N}_2)^2 < 0 \quad (17)$$

Therefore, the equilibrium state $(\overline{N}_1, \overline{N}_2)$ is globally asymptotically stable.

6. BIONOMIC EQUILIBRIUM

The term bionomic equilibrium is an amalgamation of the concepts of biological equilibrium as well as economic equilibrium. The economic equilibrium is said to be achieved when the total revenue obtained by selling the harvested biomass equals the total cost for the effort devoted to harvesting.

Let $c_2 =$ fishing cost per unit effort of the predator, $p_2 =$ price per unit biomass of the predator. The net economic revenue for the predator at any time t is given by:

ARPN Journal of Engineering and Applied Sciences

$$R_2 = (p_2 q_2 N_2 - c_2) E \tag{18}$$

The biological equilibrium is $((N_1)_\infty, (N_2)_\infty, (E)_\infty)$, where $(N_1)_\infty, (N_2)_\infty, (E)_\infty$ are the positive solutions of

$$a_1 N_1 - \alpha_{11} N_1^2 - \alpha_{12} N_1 N_2 = 0 \tag{19}$$

$$(a_2 - q_2 E) N_2 - \alpha_{22} N_2^2 + \alpha_{21} N_1 N_2 = 0 \tag{20}$$

and $(p_2 q_2 N_2 - c_2) E = 0$ (21)

From (21), we have

$$\{p_2 q_2 (N_2)_\infty - c_2\}(E)_\infty = 0 \Rightarrow (N_2)_\infty = \frac{c_2}{p_2 q_2} \tag{22}$$

From (19) and (21), we get $(N_1)_\infty = \dfrac{1}{\alpha_{11}}\left(a_1 - \alpha_{12}\dfrac{c_2}{p_2 q_2}\right)$ (23)

From (20), (22) and (23), we get

$$(E)_\infty = \frac{1}{q_2}\left[\alpha_{21}(N_1)_\infty - \left(\alpha_{22}\frac{c_2}{p_2 q_2} - a_2\right)\right] \tag{24}$$

It is clear that $(E)_\infty > 0$ if $\alpha_{21}(N_1)_\infty > \left(\alpha_{22}\dfrac{c_2}{p_2 q_2} - a_2\right)$ (25)

Thus the bionomic equilibrium $((N_1)_\infty, (N_2)_\infty, (E)_\infty)$ exists, if inequality (25) holds.

7. OPTIMAL HARVESTING POLICY

The present value *J* of a continuous time-stream of revenues is given by:

$$J = \int_0^\infty e^{-\delta t}(p_2 q_2 N_2 - c_2) E\, dt \tag{26}$$

Where δ denotes the instantaneous annual rate of discount. Our problem is to maximize *J* subject to the state equations (1) and (2) and control constraints $0 \le E \le (E)_{\max}$ by invoking Pontryagin's maximum principle.

The Hamiltonian for the problem is given by:

$$H = e^{-\delta t}(p_2 q_2 N_2 - c_2)E + \lambda_1(a_1 N_1 - \alpha_{11} N_1^2 - \alpha_{12} N_1 N_2) + \lambda_2(a_2 N_2 - \alpha_{22} N_2^2 + \alpha_{21} N_1 N_2 - q_2 E N_2) \tag{27}$$

Where λ_1, λ_2, are the adjoint variables.

Let us assume that the control constraints are not binding i.e., the optimal solution does not occur at $(E)_{\max}$. At $(E)_{\max}$ we have a singular control.

By Pontryagin's maximum principle,

$$\frac{\partial H}{\partial E} = 0;\ \frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial N_1};\ \frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial N_2}$$

$$\frac{\partial H}{\partial E} = 0 \Rightarrow e^{-\delta t}(p_2 q_2 N_2 - c_2) - \lambda_2 q_2 N_2 = 0 \Rightarrow \lambda_2 = e^{-\delta t}\left(p_2 - \frac{c_2}{q_2 N_2}\right) \tag{28}$$

$$\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial N_1} = -\{\lambda_1(a_1 - 2\alpha_{11} N_1 - \alpha_{12} N_2) + \lambda_2(\alpha_{21} N_2)\}$$

$$\Rightarrow \frac{d\lambda_1}{dt} = (\lambda_1 \alpha_{11} N_1 - \lambda_2 \alpha_{21} N_2 E) \tag{29}$$

$$\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial N_2} = -\{e^{-\delta t} p_2 q_2 E + \lambda_1(-\alpha_{12} N_1) + \lambda_2[(a_2 - q_2 E) + \alpha_{21} N_1 - 2\alpha_{22} N_2]\}$$

$$\Rightarrow \frac{d\lambda_2}{dt} = (\lambda_2 \alpha_{22} N_2 + \lambda_1 \alpha_{12} N_1 - e^{-\delta t} p_2 q_2 E) \tag{30}$$

From (28) and (29), we get $\dfrac{d\lambda_1}{dt} - \lambda_1 \alpha_{11} N_1 = -B_1 e^{-\delta t}$

Where $B_1 = \alpha_{21}\overline{N_2}\left(p_2 - \dfrac{c_2}{q_2 \overline{N_2}}\right)$

Whose solution is given by $\lambda_1 = \dfrac{B_1}{(\alpha_{11}\overline{N_1} + \delta)} e^{-\delta t}$ (31)

From (30) and (31), we get $\dfrac{d\lambda_2}{dt} - \lambda_2 \alpha_{22} N_2 = -B_2 e^{-\delta t}$

Where $B_2 = \left[p_2 q_2 E - \dfrac{B_1 \alpha_{12}\overline{N_1}}{(\alpha_{11}\overline{N_1} + \delta)}\right]$

Whose solution is given by $\lambda_2 = \dfrac{B_2}{(\alpha_{22}\overline{N_2} + \delta)} e^{-\delta t}$ (32)

From (28) and (32), we get a singular path

$$\left(p_2 - \frac{c_2}{q_2 \overline{N_2}}\right) = \frac{B_2}{(\alpha_{22}\overline{N_2} + \delta)} \tag{33}$$

Thus (33) can be written as:

$$F(\overline{N_2}) = \left(p_2 - \frac{c_2}{q_2 \overline{N_2}}\right) - \frac{B_2}{(\alpha_{22}\overline{N_2} + \delta)}$$

There exist a unique positive root $\overline{N_2} = (N_2)_\delta$ of $F(\overline{N_2}) = 0$ in the interval $0 < \overline{N_2} < k_2$, if the

VOL. 7, NO. 8, AUGUST 2012 ISSN 1819-6608
ARPN Journal of Engineering and Applied Sciences

following hold $F(0) < 0$, $F(k_2) > 0$, $F'(\overline{N_2}) > 0$ for $\overline{N_2} > 0$.

For $\overline{N_2} = (N_2)_\delta$, we get $(N_1)_\delta = \frac{1}{\alpha_{11}}\left(a_1 - \alpha_{12}\frac{c_2}{p_2 q_2}\right)$ (34)

and

$$(E)_\delta = \frac{1}{q_2}\left[\alpha_{21}(N_1)_\infty - \left(\alpha_{22}\frac{c_2}{p_2 q_2} - a_2\right)\right] \qquad (35)$$

Hence once the optimal equilibrium $\left((N_1)_\delta, (N_2)_\delta\right)$ is determined, the optimal harvesting effort $(E)_\delta$ can be determined.

From (28), (31) and (32), we found that λ_1, λ_2 do not vary with time in optimal equilibrium. Hence they remain bounded as $t \to \infty$.
From (33), we also note that

$$\left(p_2 - \frac{c_2}{q_2 \overline{N_2}}\right) = \frac{B_2}{\left(\alpha_{22}\overline{N_2} + \delta\right)} \to 0 \text{ as } \delta \to \infty$$

Thus, the net economic revenue of the predator $R_2 = 0$.

This implies that if the discount rate increases, then the net economic revenue decreases and even may tend to zero if the discount rate tend to infinity. Thus it has been concluded that high interest rate will cause high inflation rate. This conclusion was also drawn by Clark [9] in the combined harvesting of two ecologically independent populations and by Chaudhuri [10] in the combined harvesting of two competing species.

8. NUMERICAL EXAMPLES

Let $a_1 = 3; \alpha_{11} = 0.1; \alpha_{12} = 0.12; a_2 = 1.5; \alpha_{21} = 0.03; \alpha_{22} = 0.4; q_2 = 0.02 \& E = 20.$

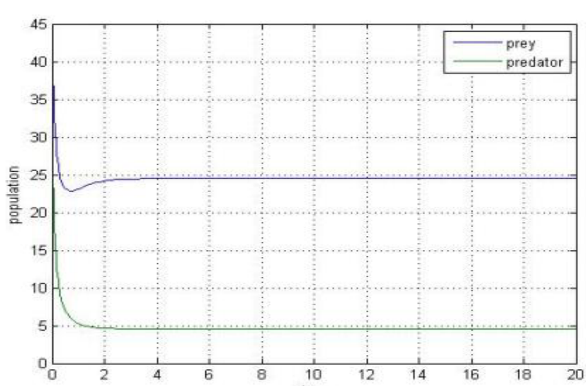

Figure-1. Shows the variation of the populations against the time.

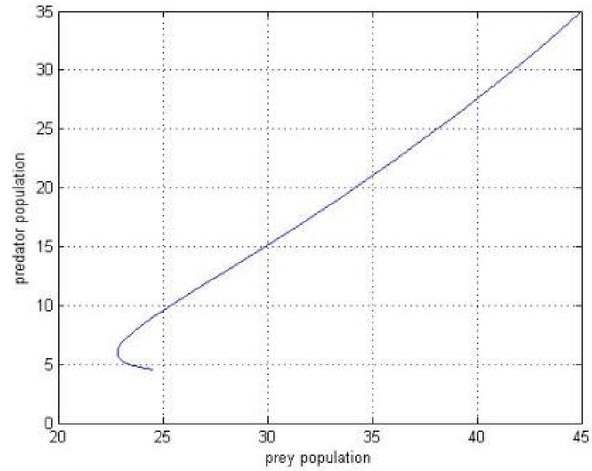

Figure-2. Shows the trajectory of the prey and predator populations beginning with $N_1 = 45$ and $N_2 = 35$.

Let $a_1 = 8; \alpha_{11} = 0.12; \alpha_{12} = 0.5; a_2 = 2; \alpha_{21} = 0.12; \alpha_{22} = 0.14; q_2 = 0.2 \& E = 15.$

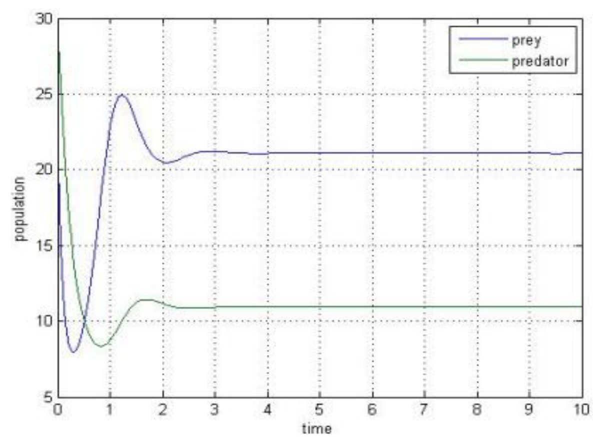

Figure-3. Shows the variation of the populations against the time.

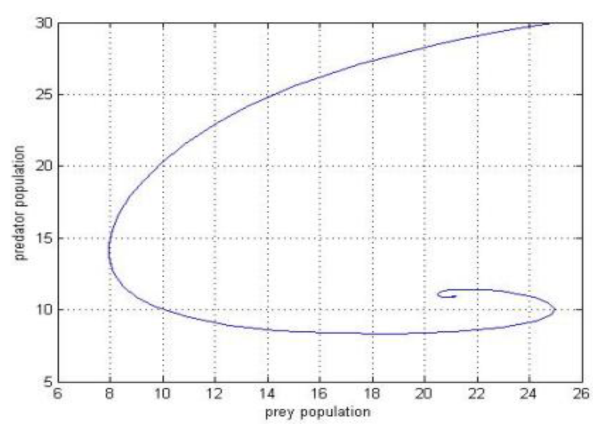

Figure-4. Shows the trajectory of the prey and predator populations beginning with $N_1 = 25$ and $N_2 = 30$.

CONCLUSIONS

A prey - predator model with an alternative food for the predator, and the predator is harvested under optimal conditions. The model is characterized by a pair of first order non-linear ordinary differential equations. All the four equilibrium points of the model are identified and the criteria for the stability were discussed by using Lyponuv's method. The bio-economic equilibrium point was identified and the optimal harvest policy was studied, and some numerical examples were discussed. Finally it was observed that the model was stabilized by harvesting.

REFERENCES

[1] Pontryagin L.S., Boltyanskii V.S., Gamkerlidge R.N. and Mischenko E.F. 1962. The mathematical theory of optimal process, Wiley, New York, USA.

[2] Lotka A.J. 1925. Elements of Physical Biology, Williams and Wilkins, Baltimore, New York, USA.

[3] Volterra V. 1931. Leçons sur la théorie mathématique de la lutte pour la vie, Gauthier-Villars, Paris, France.

[4] Paul Colinvaux A. 1986. Ecology. John Wiley and Sons Inc., New York, USA.

[5] Freedman H. I. 1980. Deterministic Mathematical Models in Population Ecology, Decker, New York, USA.

[6] Kapur J.N. 1985. Mathematical modeling in biology and Medicine. Affiliated East West Press.

[7] Kapur J.N. 1985. Mathematical modeling, wiley, easter.

[8] Clark C.W. 1985. Bioeconomic modelling and fisheries management. Wiley, New York, USA.

[9] Clark C.W. 1976. Mathematical bioeconomics: the optimal management of renewable resources. Wiley, New York, USA.

[10] Chaudhuri K.S. 1986. A bioeconomic model of harvesting a multispecies fishery. Ecol. Model. 32: 267-279.

[11] Chaudhuri K.S. 1988. Dynamic optimization of combined harvesting of a two species fishery. Ecol. Model. 41: 17-25.

[12] Chaudhuri K.S. and Saha Ray S. 1996. On the combined harvesting of a prey predator system. J. Biol. Sits. pp. 376-389.

[13] Kar T.K. and ChaudhuriK. S. 2004. Harvesting in a two-prey one-predator fishery: A bioeconomic model. ANZIAM J. 45: 443-456.

[14] ArchanaReddy R., Pattabhiramacharyulu N.Ch and KrishnaGandhi B. 2007. A stability analysis of two competitive interacting species with harvesting of both the species at a constant rate. Int. J. of scientific computing. (1) Jan- June. pp. 57-68.

[15] LakshmiNarayan K. 2005. A mathematical study of a prey-predator ecological model with a partial cover for the prey and alternative food for the predator, Ph.D thesis, JNTU.

[16] Lakshmi Narayan K. and Pattabhiramacharyulu N. Ch. 2008. A Prey-Predator Model with an Alternative Food for the Predator, Harvesting of Both the Species and with a Gestation Period for Interaction. Int. J. Open Problems Compt. Math. 1: 71-79.

[17] Shiva Reddy K., Lakshmi Narayan K. and PattabhiramacharyuluN. Ch. 2010. A Three Species Ecosystem Consisting of a Prey and Two Predators. International J. of Math. Sci. and Engg. Appls. 4(IV): 129-145.