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APPLICATION OF DIRECT VARIATIONAL METHOD IN THE ANALYSIS OF ISOTROPIC THIN RECTANGULAR PLATES

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ABSTRACT

The popular methods of analysis of thin rectangular plates have been numerical and classical procedures. These methods, especially the classical method have always been tedious and rigorous. In this study, the mathematical model, that is based on direct variational procedures and potential energy principle, is developed and successfully applied to: (i) Thin rectangular Plates with two opposite edges clamped and other opposite edges simply supported and (ii) Thin rectangular plates with one edge clamped and the three other sides simply supported. The coordinate functions, which must satisfy the geometric and natural boundary conditions, are carefully constructed and applied into classical plate equation. The plate equation is thus integrated and the integrand minimized to obtain the unknown coefficients which when substituted back in deformation equation of mid-surface of plate gives the deformation surface of plate in analytical form. This enables the evaluation of deflections and bending moments at any arbitrary point on the plates unlike the numerical methods which only give these results at nodal points. The results obtained from this study have excellent comparison with those of numerical and classical solutions obtained from literature. The study also clearly shows that direct variational method circumvents the tedious and rigorous procedures involved in the classical and numerical methods.

Keywords: bending moment, deflections, direct variational method, energy principle, thin rectangular plate.

1. INTRODUCTION

Ventsel and Krauthemmer (2001) classified plates into thick, membranes and thin plates. To be considered in this research work is thin rectangular plates which are intermediates between thick and membrane plates. Plates may be classified as isotropic or orthotropic. Isotropic plates refer to plates whose material properties in all directions at a point are same while anisotropic or orthotropic plates refer to plates whose material properties are direction dependent. The predominant transverse loads on the plates are static and dynamic in nature. In this study, isotropic thin rectangular plates with static transverse loads are considered. Due to prevailing and frequent uses of plates in structural, mechanical and aeronautical Engineering; a lot of researches are being carried out on plates. Plates are predominantly used in engineering due to its light weight, economy and its ability to withstand heavy loads. The loads on the plate could be uniformly distributed, partially distributed or concentrated loads.

The support conditions of plates may be different on each side of four sided plate. A pair of parallel sides may be simply supported and other two sides clamped or two adjacent sides may be clamped with other two sides simply supported or free. In the present study, investigations are to be carried out on plates with two boundary conditions: (i) Plate with two opposite sides simply supported and the other opposite sides clamped. (ii) Plate with two adjacent sides clamped and other adjacent sides simply supported.

Before now, the common method of analysis has been classical solution using either trignometrical or double series. This was followed by numerical methods like Finite Element Method, Boundary Element Method, Finite Strip Method, Grid work Method, Finite Difference Methods etc. Dey (1981) researched on the bending and deflection analysis of rectangular plate using a combination of basic functions and Finite Difference energy technique in what is called "Semi-numerical Analysis of Rectangular Plates in Bending". Gierlinski and Smith (1984) utilized Finite Strip approach to determine the Geometric non-linear analysis of thin walled structures. The theory used is based on moderately large displacement assumptions giving non-linear strain-displacement relations but linear curvature-displacement relations. Mbakogu and Pavlovic (1988) applied algebra to the classical problems in plate theory. Based on literature survey conducted, little analytical work is done on plate using direct variational procedures to solve the plate bending problems. Taylor and Govindjee (2002) utilized double cosine series expansion and exploitation of the Sherman-Morrison-woodbury formula. Zenkour (2003) in his works on "Exact Mixed-Classical Solution for the Bending Analysis of Shear deformable Rectangular Plates" discovered that thin plate model does not provide a very good analysis of plates in which the thickness-to-length ratio is relatively large. The method is very difficult but accurate. Hasebe and Wang (2002) also applied Green functions for the bending of thin plates under various boundary conditions and applications. The application of Green's function by Hasebe and Wang investigated the interaction of a hole or inclusion with a crack and the interaction of the debonded inter-surface with a crack

This work intends to utilize the Direct variational method as formulated by Ritz to solve plates: (i) Thin Rectangular plates with 2 opposite sides fixed and other opposite sides simply supported (ii) Thin Rectangular plates with 2 adjacent sides clamped and other adjacent sides simply supported.

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2. FORMULATION OF PLATE EQUATION USING ENERGY PRINCIPLE

The general equation of plate using total potential Energy principles consists of strain Energy of deformation U and potential Energy of External work we, assuming the element of the structure under the transverse load remains elastic and is under adiabatic condition. Obeying strictly Hooks law, the Strain Energy of the plate is:

$$\vartheta = \frac{1}{2}\int \sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \tau_{xy}\gamma_{xy} du \quad - \quad (1)$$

Where σ_x = normal stress along the x-axis

σ_y = normal stress along the y-axis

τ_{xy} = shear stress along the x-y plane.

$\varepsilon_x, \varepsilon_y$ and γ_{xy} are the respective strains on x, y, axes and x-y plane.

Where E = modulus of elasticity

ϑ = Poisson ratio

The Strain Energy U can be written in terms of curvature by substituting the values of stresses and Strains of equations 2(a-c) and 3 (a-c) into equation (1) and simplifying to obtain

$$U = \frac{1}{2}\int \frac{\varepsilon z^2}{\left(1-\vartheta^2\right)} dz \iint_A \left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2\vartheta \frac{\partial^2 w}{\partial x^2}.\frac{\partial^2 w}{\partial y^2} + 2\left(1-\vartheta\right)\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right] dxdy \quad \_ \quad (2)$$

Integrating the first term of Equation (2) over the entire thickness of the surface from $-h/2$ to $h/2$ and simplifying, we obtain

$$U = \frac{D}{2}\iint \left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2\vartheta \frac{\partial^2 w}{\partial x^2}.\frac{\partial^2 w}{\partial y^2} + 2\left(1-\vartheta\right)\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right] dxdy \quad \_ \quad (3)$$

Where D = Flexural Rigidity = $\dfrac{Eh^3}{12(1-\vartheta^2)}$

In the present study, the plate is acted upon by uniformly distributed transverse load. Therefore, the external work We =

$$\int q\, w(x,y)\, dxdy. \quad - \quad (6)$$

Therefore total potential Energy

$$= U - We_{xt} \quad - \quad (7a)$$

$$= \frac{D}{2}\iint \left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2\vartheta\left(\frac{\partial^2 w}{\partial x^2}.\frac{\partial^2 w}{\partial y^2}\right) + 2\left(1-\vartheta\right)\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - q\right] dxdy \quad \_ \quad (7b)$$

3. METHODOLOGY

The Modified Direct variational method of Ritz is adopted here. Apart from satisfying geometric Boundary conditions, the natural boundary conditions are deemed to be satisfied. It is based on principle of minimum total potential Energy.

The total potential Energy of plate from equation 7(b) is:

$$\Pi = \iint \left\{\frac{D}{2}\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2\vartheta \frac{\partial^2 w}{\partial x^2}.\frac{\partial^2 w}{\partial y^2} + 2\left(1-\vartheta\right)\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - qW(x,y)\right\} \partial x \partial y \quad \_ \quad (4)$$

Where W (x,y) is the plates deformation surface which is being approximated in this study as a n-term variable – Separable polynomial as:

$$= C_1\phi_1(x)\varphi_1(y) + C_2\phi_2(x)\varphi_2(y) + C_3\phi_3(x).\varphi_3(y) \ldots + C_n\phi_n(x).\varphi_n(y) \quad - \quad (5)$$

where $\phi_1, \phi_2, \phi_3 \ldots \phi_n$, *and* $\varphi_1, \varphi_2, \varphi_3 \ldots \varphi_n$ *are constructed co – ordinate functions in x and y axes respectively.*

$\varphi_i(y)$ *is derviable from* $\phi_i(x)$ *by replacing x by y and a by b*

Equation (5) could be simplified further by putting

$$h_1 = \phi_1(x).\varphi_1(y),$$
$$h_2 = \phi_2(x).\varphi_2(y)$$
$$h_3 = \phi_3(x).\varphi_3(y)$$

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$$h_n = \phi_n(x).\varphi_n(y) - \quad (6)$$

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Substituting equation (2) into equation (8), the deformation surface of the plate could now be written as:

$$W(x,y) = C_1 h_1 + C_2 h_2 + C_3 h_3 + ... + C_n h_n - (7a)$$

$$W(x,y) = HC \qquad - \qquad (7b)$$

Where H = [h_1 h_2 h_3 h_4]

C = [C_1 C_2 C_3 C4].

The functions of H polynomial of equation 7(b) must satisfy the kinematic boundary conditions and are linearly independent and continuous. These functions of equation (7) are subsequently substituted into the total potential Energy equation of (4) above and on simplifying after matrix multiplication rule, we obtain:

$$\prod = \iint \frac{D}{2} C^T H_{xx}{}^T . H_{xx} C + C^T . H_{yy}{}^T . H_{yy} . C + 2\vartheta C^T H_{xx} . H_{yy} C + 2(1-\vartheta) C^T . H_{xy}{}^T H_{xy} C - C^T H^T q \} dxdy \qquad - \qquad (8)$$

$$\prod = C^T \left(\frac{D}{2} \left(A_1 + A_2 + 2\vartheta A_3 + 2(1-\vartheta) A_4 \left[C \right] - (9) \right. \right.$$

$$Where\ A_1 = \iint_A H_{xx}{}^T . H_{xx}\ dxdy$$

$$A_2 = \iint_A H_{yy}{}^T . H_{yy}\ dxdy$$

$$A_3 = \iint_A H_{xx} . H_{yy}\ dxdy$$

$$A_4 = \iint_A H_{xy}{}^T . H_{xy}\ dxdy$$

$$B = \iint_A H^T .\ dxdy$$

For the Equilibrium condition of the plate under the transverse loading to be maintained, the total potential Energy $\prod$ will be minimized.

$$ie \qquad \frac{\partial \prod}{\partial C_i} = 0,\ i = 1, 2, 3, ...n \qquad - \qquad (10)$$

$$\frac{\partial \prod}{\partial C^T} = \frac{D}{2} \left[A_1 + A_2 + 2\vartheta A_3 + 2(1-\vartheta) A_4 \right] \left[C \right] = q \left[B \right] \qquad - \qquad (11a)$$

$$\frac{D}{2} \left[A_1 + A_2 + 2\vartheta A_3 + (2-\vartheta) A_4 \right\} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ \vdots \\ C_n \end{bmatrix} = \begin{bmatrix} q_1{}^x \\ q_2{}^x \\ q_3{}^x \\ . \\ . \\ . \\ q_n{}^x \end{bmatrix} - (11b)$$

$$where\ q_i{}^x = qb_j \qquad - \qquad (12)$$

$$and\ B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \qquad - \qquad (14d)$$

On evaluation of the unknown coefficients, C_1, C_2, C_3, and C_4 from the simultaneous equation (14b), the coefficients are substituted into equation (10) to obtain the deformation surface of the plate in analytical form Subsequently the deflection and moments on any arbitrary point on the plate can be obtained using the following equations**.**

$$W(x,y) = HC \qquad - \qquad (13a)$$

$$M_x = -D \left(\frac{\partial^2 w}{\partial x^2} + \vartheta \frac{\partial^2 w}{\partial y^2} \right) \qquad - \qquad (13b)$$

$$M_y = -D \left(\frac{\partial^2 w}{\partial y^2} + \vartheta \frac{\partial^2 w}{\partial x^2} \right) \qquad - \qquad (13c)$$

$$M_{xy} = -2D(1-\vartheta) \frac{\partial^2 w}{\partial x \partial y}. \qquad - \qquad (13d)$$

4. ANALYSIS OF PLATES AND THE RESULTS

4.1 Thin rectangular plate with two opposite edges simply supported and other two opposite edges clamped under uniformly distributed load

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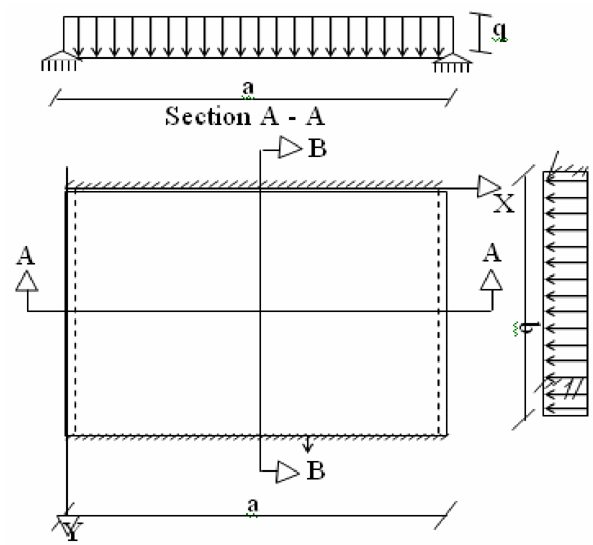

Figure-1. Thin rectangular plate with two (2) edges simply supported and two (2) edges clamped and subjected to uniform distributed load.

The boundary conditions for plates with two opposite edges simply supported and the other two edges clamped are:

$$w(x) = \frac{\partial^2 w}{\partial x^2}(x)\, at\, x = 0, a \qquad - \quad (14a)$$

$$w(y) = \frac{\partial w}{\partial x}(y)\, at\, y = 0, b \qquad - \quad (14b)$$

The constructed co-ordinate functions that satisfy the above boundary conditions are:

$$\phi_1(x) = \phi_2(x) = \left(\frac{x}{a}\right) - 2\left(\frac{x}{a}\right)^3 + \left(\frac{x}{a}\right)^4 \quad -(15a)$$

$$\phi_3(x) = \phi_2(x) = 3\left(\frac{x}{a}\right) - 5\left(\frac{x}{a}\right)^3 + 3\left(\frac{x}{a}\right)^5 - \left(\frac{x}{a}\right)^6 - (15b)$$

Similarly, the respective constructed co-ordinate functions in the y-axis are:

$$\varphi_1(y) = \varphi_2(y) = \left(\frac{y}{b}\right)^2 - 2\left(\frac{y}{b}\right)^3 + \left(\frac{y}{b}\right)^4 \qquad -(15c)$$

$$\varphi_3(y) = \varphi_4(y) = 3\left(\frac{y}{b}\right)^2 - 5\left(\frac{y}{b}\right)^3 + 3\left(\frac{y}{b}\right)^5 - \left(\frac{y}{b}\right)^6 (15d)$$

The deformation mid surface of the plate is represented by:

$$w(x, y) = \sum_{i=1}^{n} c_i \phi_i(x) \varphi_i(y) \qquad - (16)$$

The above equation is however broken down respectively into one-term, two- term, three-term and four-term polynomials as:

$$w(x, y) = c_1\phi_1(x).\varphi_1(y) \qquad -(16a)$$

$$w(x, y) = c_1\phi_1(x).\varphi_1(y) + c_2\phi_2(x).\varphi_2(y) \qquad - (16b)$$

$$w(x, y) = c_1\phi_1(x).\varphi_1(y) + c_2\phi_2(x).\varphi_2(y) + c_3\phi_3(x).\varphi_3(y) \qquad -(16c)$$

$$w(x, y) = c_1\phi_1(x).\varphi_1(y) + c_2\phi_2(x).\varphi_2(y) + c_3\phi_3(x).\varphi_3(y) + c_4\phi_4(x).\varphi_4(y) \qquad -(16d)$$

Where $c_{1,}, c_2, c_3, and\ c_4$ are the unknown coefficients to be determined, while $\phi_1(x), \phi_2(x), \phi_3(x), \phi_4(x)\, and\, \varphi_1(y), \varphi_2(y), \varphi_3(y), \varphi_4(y)$ are as represented in equations 15(a-d) above.

The one-term polynomial of equation (16a) is substituted into equation (8) via equation (9). The potential equation (9) is integrated and the integrand subsequently minimized to obtain the unknown coefficient, c_1. The determined coefficient is however substituted back into the equation of deformation surface of plate to obtain the deflection of the plate in analytical form. The maximum deflection and moments at the mid-point and other arbitrary points are then obtained at various plate aspect ratios. The same evaluations is repeated respectively using two-term, three-term and four-term polynomials as represented in equations 16(b-d) and the results obtained are presented in Tables 1 to 4.

4.2 Thin rectangular plate with 3 edges simply supported and the other edge clamped under uniformly distributed load.

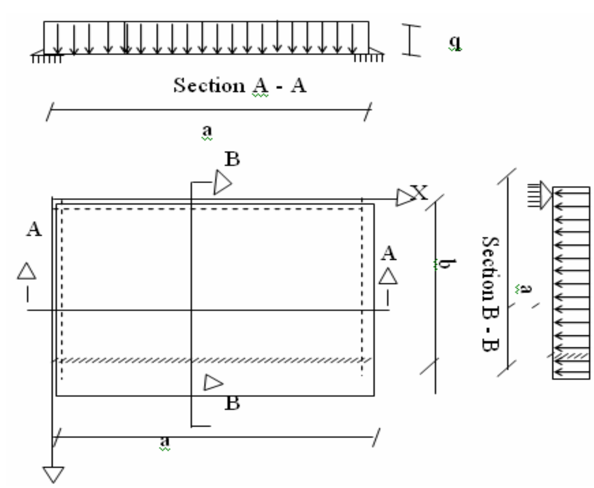

Figure-2. Thin Rectangular plate with 3 edges simply supported and the other edge fixed and subjected to uniformly distributed load.

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The boundary conditions for plate with 3 edges simply supported and the fourth edge clamped are:

$$w(x) = \frac{\partial^2 w}{\partial x^2}(x) = 0 \, at \, x = 0, a \qquad -(17a)$$

$$w(y) = \frac{\partial^2 w}{\partial y^2}(y) = 0 \, at \, y = 0 \qquad -(17b)$$

$$w(y) = \frac{\partial w}{\partial y}(y) = 0 \, at \, y = b \qquad -(17c)$$

The algebraic functions which are continuous and at the same time satisfy the geometric and natural boundary conditions of equations 17(a-c) above are:

$$\phi_1(x) = \phi_2(x) = \left(\frac{x}{a}\right) - 2\left(\frac{x}{a}\right)^3 + \left(\frac{x}{a}\right)^4 \qquad -(18a)$$

$$\phi_3(x) = \phi_2(x) = 3\left(\frac{x}{a}\right) - 5\left(\frac{x}{a}\right)^3 + 3\left(\frac{x}{a}\right)^5 - \left(\frac{x}{a}\right)^6 \qquad (18b)$$

$$\varphi_1(y) = \varphi_2(y) = \left(\frac{y}{b}\right)^2 - 2\left(\frac{y}{b}\right)^3 + \left(\frac{y}{b}\right)^4 \qquad -(18c)$$

$$\varphi_3(y) = \varphi_4(y) = 3\left(\frac{y}{b}\right)^2 - 5\left(\frac{y}{b}\right)^3 + 3\left(\frac{y}{b}\right)^5 - \left(\frac{y}{b}\right)^6 \qquad (18d)$$

Similarly, the algebraic equations representing the deformation surface of the plate is represented as in equation (16) as:

$$w(x, y) = \sum_{i=1}^{n} c_i \phi_i(x) \varphi_i(y) \qquad - \qquad (19)$$

This could be represented in the form of one-term, two-term, three-term and four-term polynomial as in equations 16(a-d).

$$w(x, y) = c_1\phi_1(x).\varphi_1(y) + c_2\phi_2(x).\varphi_2(y) \qquad -(19b)$$

$$w(x, y) = c_1\phi_1(x).\varphi_1(y) + c_2\phi_2(x).\varphi_2(y) + c_3\phi_3(x).\varphi_3(y) \qquad - \qquad (19c)$$

$$w(x, y) = c_1\phi_1(x).\varphi_1(y) + c_2\phi_2(x).\varphi_2(y) + c_3\phi_3(x).\varphi_3(y) + c_4\phi_4(x).\varphi_4(y) \qquad -(19d)$$

$c_{1,}, c_2, c_3, and \, c_4$ are the unknown coefficients of deformation surface of plate while $\phi_1(x), \phi_2(x), \phi_3(x), \phi_4(x) \, and \, \varphi_1(y), \varphi_2(y), \varphi_3(y), \varphi_4(y)$ are the constructed co-ordinate functions as represented in equations 18(a-d) above. Similar evaluations are performed as in section 4.1 using one-term, two-term, three-term and four-term polynomials of the equations 19(a-d) The results of deflections and moments obtained for various plate aspect ratios are presented on Tables 1 to 3. On application of uniformly distributed load on plate with three edges simply supported and the other edge clamped as shown in Figure-2 above, the respective values of equations 19(a-d) are substituted into equations 7(a-b). Subsequently, the result obtained is substituted into equation (4). The mathematical expression is further integrated and the integrand minimized and solved to determine the coefficients. The coefficients are respectively substituted back into equation 9(a-d) to obtain the approximate deformation surface of the plate in analytical form.

Consequently, the deflection and moments of the isotropic rectangular plate with 3 edges simply supported and the other edge clamped are obtained for various plate aspect ratios using equations 13(a-c) and 13(d), respectively. These results are presented on Tables 6 through 8.

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Table-1. Maximum deflection coefficients (α) in isotropic thin rectangular plate with two opposite edges clamped and the other opposite edges simply supported and subjected to uniformly distributed load for various plate aspect ratios ($\upsilon = 0.30$).

Deflection (W_{max})= $\alpha \frac{qa^4}{D}$, α at x = a/2, y = b/2					
Span ratio = (b/a)	**Classical method**	**Present study**			
		1 term of h	**2 terms of h**	**3 terms of h**	**4 terms of h**
1.0	0.00192	0.00199(3.65%)	0.00196(2.08%)	0.00191(-0.52%)	0.00191(-0.52%)
1.1	0.00251	0.00261(3.98%)	0.00257(2.39%)	0.00252(0.40%)	0.00252(0.40%)
1.2	0.00319	0.00330(3.45%)	0.00323(1.25%)	0.00319(nil)	0.00319(nil)
1.3	0.00388	0.00402(3.61%)	0.00393(1.29%)	0.00389(0.26%)	0.00389(0.26%)
1.4	0.00460	0.00477(3.70%)	0.00464(0.87%)	0.00460(nil)	0.00460(nil)
1.5	0.00531	0.00551(3.7%)	0.00535(0.75%)	0.00531(nil)	0.00531(nil)
1.6	0.00603	0.00624(3.48%)	0.00604(0.16%)	0.0060(-0.50%)	0.0060(-0.50%)
1.7	0.00668	0.00695(4.04%)	0.00670(0.30%)	0.00666(-0.30%)	0.00666(-0.30%)
1.8	0.00732	0.00762(3.93%)	0.00732(nil)	0.00728(nil)	0.00732(nil)
1.9	0.00790	0.00826(4.56%)	0.00790(nil)	0.00787(-0.40%)	0.00787(-0.40%)
2.0	0.00844	0.00885(4.86%)	0.00843(-0.12%)	0.00840(-0.47%)	0.00840(-0.47%)
3.0	0.01168	0.01281(9.67%)	0.01148(-1.71%)	0.01147(1.80%)	0.01147(1.80%)

- The values in the bracket indicate the % variation of the present study from the classical solution

Table-2. Maximum short span moment's coefficients (β) in a thin rectangular plate with two opposite edges clamped and the other opposite edges simply supported and subjected to uniformly distributed load ($\upsilon = 0.30$).

Short span moment (M_{max}) = βqa^2 ; β , at x = a/2, y = b/2					
Span ratio = (b/a)	**Classical method**	**Present study**			
		1 term of h	**2 terms of h**	**3 terms of h**	**4 terms of h**
1.0	0.0244	0.0286(17.21%)	0.0274(12.30%)	0.0238(-2.0%)	0.0239(-2.0%)
1.1	0.0307	0.0355(15.64%)	0.0338(10.09%)	0.0302(-1.65%)	0.0305(-0.6%)
1.2	0.0376	0.0427(13.56%)	0.0404(7.45%)	0.0370(-1.60%)	0.0371(1.3%)
1.3	0.0446	0.0501(12.33%)	0.0473(6.05%)	0.0439(9.86%)	0.0440(-1.34%)
1.4	0.0514	0.0575(11.88%)	0.0540(5.06%)	0.0508(-1.17%)	0.0509(-0.97%)
1.5	0.0585	0.0647(10.60%)	0.0605(3.42%)	0.0575(-1.7%)	0.0572(-2.22%)
1.6	0.065	0.0716(10.15%)	0.0664(2.15%)	0.0639(-1.69%)	0.0642(-1.23%)
1.7	0.0712	0.0783(9.17%)	0.0726(1.97%)	0.0699(-1.82%)	0.0700(-1.69%)
1.8	0.0768	0.0845(10.03%)	0.0780(1.56%)	0.0755(-1.67%)	0.0783(1.95%)
1.9	0.0821	0.0903(9.99%)	0.0830(1.10%)	0.0806(-1.80%)	0.0806(-1.80%)
2.0	0.0844	0.0956(13.27%)	0.08743.55%)	0.0852(-0.95%)	0.0859(-1.78%)
3.0	0.1144	0.1209(6.68%)	0.1116(-2.46%)	0.1105(-3.41%)	0.1111(-2.88%)

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Table-3. Maximum long span moments coefficients (β) in a thin rectangular plate with 2 opposite edges simply supported and the other 2 opposite edges clamped and subjected to uniformly distributed load (υ = 0.30).

Long span moment (M_{max}) = $\beta_1 qa^2$: β_1, at x = a/2, y = b/2					
Span ratio = (b/a)	**Classical method**	**Present study**			
		1 term of h	**2 terms of h**	**3 terms of h**	**4 terms of h**
1.0	0.0332	0.0375(12.95%)	0.0343(3.31%)	0.0326(-1.81%)	0.0327(-1.5%)
1.1	0.0371	0.0421(13.48%)	0.0378(2.96%)	0.0362(-2.43%)	0.0365(-1.62%)
1.2	0.0400	0.0462(15.5%)	0.0407(1.75%)	0.0392(-2.0%)	0.0395(-1.25%)
1.3	0.0426	0.0497(16.67%)	0.0431(1.17%)	00417(-2.11%)	0.0417(-2.11%)
1.4	0.0448	0.0527(17.63%)	0.0448(nil)	0.0435(-2.90%)	0.0436(-2.67%)
1.5	0.0460	0.0551(19.78%)	0.0460(nil)	0.0448(2.61%)	0.0445(-3.2%)
1.6	0.0469	0.0570(21.54%)	0.0467(-0.43%)	0.0456(-2.77%)	0.0458(-2.35%)
1.7	0.0475	0.0585(23.16%)	0.0469(-1.28%)	0.0459(-3.37%)	0.0460(-3.16%)
1.8	0.0477	0.0596(24.95%)	0.0469(-1.68%)	0.0459(-3.77%)	0.0477(nil)
1.9	0.0476	0.0604(26.89%)	0.0465(-2.31%)	0.0457(-4.01%)	0.0457(-4.01%)
2.0	0.0474	0.0609(28.48%)	0.0460(-2.95%)	0.0452(-4.64%)	0.0456(-3.6%)
3.0	0.0419	0.597(42.48%)	0.0377(-10%)	0.0375(-10.02%)	0.0421(0.47%)

- The values in the bracket indicate the % variation of the present study from the classical solution

Table-4. Maximum long span edge moments coefficients (β'') in a thin rectangular plate with 2 opposite edges simply supported and the other 2 opposite edges clamped and subjected to uniformly distributed load (υ = 0.30).

Long span moment (M_{max}) = $\beta'' qa^2$, : β'' at x = a/2, y = b/2					
Span ratio = (b/a)	**Classical method**	**Present study**			
		1 term of h	**2 terms of h**	**3 terms of h**	**4 terms of h**
1.0	-0.0697	-0.0636(-8.75%)	-0.0733(5.16%)	-0.0719(3.16%)	-0.0716(2.72%)
1.1	-0.0787	-0.0691(-12.20%)	-0.0820(4.19%)	-0.0808(2.67%)	-0.0801(1.78%)
1.2	-0.0868	-0.0733(-15.55%)	-0.0896(3.23%)	-0.0887(2.19%)	-0.0885(1.96%)
1.3	-0.0938	-0.0762(-18.76%)	-0.0962(2.49%)	-0.0954(1.71%)	-0.0953(-1.60%)
1.4	-0.0998	-0.0799(-19.94%)	-0.1017(1.90%)	-0.1010(1.20%)	-0.1008(1.0%)
1.5	-0.1049	-0.0784(-25.26%)	-0.1061(-2.76%)	-0.1055(0.57%)	-0.1061(1.14%)
1.6	-0.1090	-0.0780(-28.40%)	-0.1095(0.45%)	-0.1090(nil)	-0.1086(-0.36%)
1.7	-0.1122	-0.0770(-31.37%)	-0.1121(0.09%)	-0.1116(-0.53%)	-0.1115(-0.62%)
1.8	-0.1152	-0.0753(-34.64%)	-0.1138-1.21%)	-0.1135(-1.48%)	-0.1135(-1.48%)
1.9	-0.1174	-0.0732(-41.21%)	-0.1149(-2.13%)	-0.1146(-2.39%)	-0.1145(-2.47%)
2.0	-0.1191	-0.0708(-40.55%)	-0.1154(-3.11%)	-0.1151(3.36%)	-0.1145(-3.86%)

Table-5. Comparison of finite difference method, classical solution and present study for square rectangular plate with two opposite edges clamped and other two edges simply supported under a Uniformly distributed load ($\upsilon = 0.3$).

Matrix size	Solution method	$W_{max} = \alpha \frac{pa^4}{D}$ α	$Mx_{max(span)} = \beta qa^2$ β	$My_{(span)} = \beta_1 qa^2$ β_1	$My_{max}(edge) = \beta^{11} qa^2$ β^{11}
4 x 4	Finite difference	0.00247 (28.6%)	0.02896(18.6%)	0.03344 (0.7%)	-05018 (-25%)
8 x 8		0.002088 (8.3%)	0.02586 (6.0%)	0.03338 (0.5%)	-0.06489 (-7.0%)
	Classical method	0.00192	0.0244	0.0332	-0.0697
1 x 1	Present study	0.00199(3.65%)	0.0286(17.21%)	0.0375(12.95%)	-0.0636(-8.75%)
2 x 2		0.00196(2.08%)	0.0274(12.30%)	0.0343(3.31%)	-0.0733(5.16%)
3 x 3		0.00191(-0.52%)	0.0238(2.0%)	0.0326(-1.81%)	-0.0719(3.16%)
4 x 4		0.00191 (-0.52%)	0.0239 (-2.05%)	0.0327 (-1.5%)	-0.0716 (2.7%)

- The values in the bracket indicate the % variation of the present study from the classical solution
- (source- Aginam, 2011)

Table-6. Maximum deflection coefficients (α) in isotropic thin rectangular plate with 3 edges simply supported and the other edge clamped and subjected to uniformly distributed load for various plate aspect ratios (υ =0.30).

Deflection (W_{max}) = $\alpha \frac{qa^4}{D}$ α, at x = a/2, y = b/2					
Span ratio = (b/a)	Classical method	Present study			
		1 term of h	2 terms of h	3 terms of h	4 terms of h
0.5	0.00031	0.00033	0.00033	0.00030	0.00030
1/1.5	0.00083	0.00087	0.00087	0.00084	0.00084
1/1.4	0.00104	0.00109	0.00108	0.00105	0.00105
1/1.3	0.00133	0.00136	0.00136	0.00132	0.00132
1/1.2	0.00168	0.00172	0.00172	0.00168	0.00168
1/1.1	0.00218	0.00220	0.00219	0.0021	0.00215
1.0	0.0028	0.00282(0.71%)	0.00281(0.36%)	0.00276(-1.43%)	0.00276(-1.43%)
1.1	0.0035	0.00354(1.14%)	0.00352(0.57%)	0.00348(-0.57%)	0.00348(-0.57%)
1.2	0.0043	0.00429(-0.23%)	0.00426(-0.94%)	0.00422(-1.36%)	0.00422(-1.36%)
1.3	0.0050	0.00503(0.60%)	0.00499(-0.2%)	0.00495(-1%)	0.00495(-1.0%)
1.4	0.0058	0.00576(-0.69%)	0.00571(-1.55%)	0.00567(-2.24%)	0.00567(-2.24%)
1.5	0.0064	0.00646(0.94%)	0.00639(-0.16%)	0.00635(-0.78%)	0.00635(-0.78%)
1.6	-	0.00713	0.00703	0.00700	0.0070
1.8	-	0.00833	0.00819	0.00816	0.00816
1.9	-	0.00887	0.00870	0.00867	0.00867
2.0	0.0093	0.00937(0.75%)	0.00917(-1.40%)	0.00914(-1.75%)	0.00917(-1.40%)

- The values in the bracket indicate the % variation of the present study from the classical solution

Table-7. Maximum short moment coefficient (β) in isotropic thin rectangular plate with three edges simply supported and one edge clamped and subjected to uniformly distributed load (υ = 0.30).

Short span moment(Mxx_{max}) = βqa^2 β, at x = a/2, y = b/2					
Span ratio = (b/a)	**Classical method**	**Present study**			
		1 term of h	**2 terms of h**	**3 terms of h**	**4 terms of h**
0.5	0.00575	0.0079	0.0079	0.0056	0.0056(-2.61%)
1/1.5	0.0124	0.0154	0.0154	0.0125	0.0125(0.81%)
1/1.4	0.0153	0.0181	0.0180	0.0150	0.0150(-1.96%)
1/1.3	0.0183	0.0214	0.0213	0.0182	0.0181(-1.09%)
1/1.2	0.0222	0.0255	0.0253	0.0222	0.0220(-0.90%)
1/1.1	0.0273	0.0307	0.0304	0.0273	0.0271(-0.73%)
1.0	0.0340	0.0372(9.41%)	0.0369(8.5%)	0.0337(-0.88%)	0.0336(-1.18%)
1.1	0.041	0.0445(8.54%)	0.0441(7.56%)	0.0410(nil)	0.0410(nil)
1.2	0.049	0.0519(5.92%)	0.0512(4.49%)	0.0482(-1.42%)	0.0483(-1.42%)
1.3	0.056	0.0590(5.36%)	0.0582(3.93%)	0.0553(-1.25%)	0.0553(-1.25%)
1.4	0.063	0.0659(4.60%)	0.0648(2.86%)	0.0621(-1.43%)	0.0621(-1.43%)
1.5	0.069	0.0724(4.93%)	0.0710(2.90%)	0.0685(-0.72%)	0.0685(-0.72%)
1.6	-	0.784	0.0768	0.0744	0.0743(n/a)
1.8	-	0.0892	0.0870	0.0849	0.0850(n/a)
1.9	-	0.0940	0.0914	0.0894	0.0894(n/a)
2.0	0.094	0.0984(4.68%)	0.0954(1.49%)	0.0935(-0.21%)	0.0938(-0.21%)

- The value in the bracket indicates the % variation of the present study from the classical solution

Table-8. Maximum long moment coefficient (β_1) in isotropic thin rectangular plate with three edges simply supported and one edge clamped and subjected to uniformly distributed load (υ = 0.30).

Long span moment (Mxx_{max}) = $\beta_1 qa^2, \beta_1$, at x = a/2, y = b/2					
Span ratio = (b/a)	**Classical method**	**Present study**			
		1 term of h	**2 terms of h**	**3 terms of h**	**4 terms of h**
0.5	0.015	0.0167	0.0166	0.0149	0.0148
1/1.5	0.0240	0.0261	0.0259	0.0242	0.0242
1/1.4	0.0265	0.0287	0.0282	0.0267	0.0267
1/1.3	0.0296	0.0316	0.0313	0.0296	0.0296
1/1.2	0.0326	0.0348	0.0344	0.0328	0.0326
1/1.1	0.0355	0.0382	0.0377	0.0362	0.0361
1.0	0.039	0.0419(7.44%)	0.0412(5.6%)	0.0398(2.05%)	0.0397(1.79%)
1.1	0.042	0.0453(7.86%)	0.0443(5.48%)	0.0430(2.38%)	0.0430(2.38%)
1.2	0.044	0.0481(9.32%)	0.0467(6.28%)	0.0455(3.41%)	0.0455(3.41%)
1.3	0.045	0.0502(11.56%)	0.0486(8%)	0.0475(5.55%)	0.0474(5.33%)
1.4	0.047	0.0519(10.42%)	0.0499(6.17%)	0.0488(3.83%)	0.0487(3.62%)
1.5	0.048	0.0531	0.0507	0.0498(3.75%)	0.0498(3.75%)
1.6	-	0.0539	0.0512	0.0503	0.0503(n/a)
1.8	-	0.0549	0.0514	0.0507	0.0508(n/a)
1.9	-	0.0550	0.0513	0.0506	0.0506(n/a)
2.0	0.047	0.0551	0.0510	0.0504	0.0505(7.45%)

- The values in the bracket indicate the % variation of the present study from the classical solution

5. DISCUSSION OF RESULTS

5.1 Thin rectangular plate with 2 opposite edges simply supported and the other opposite edges clamped

Table-1 compares the maximum deflection coefficient (α) of the study with the classical solution (Timoshenko and Woinosky-Kreiger, 1959). The results obtained from the study show satisfactory agreement with the classical solution. The accuracy of the results improves as the number of terms in the polynomials increase, producing better results at the three and four term polynomials. The percentage variation of the results of this study with the classical solution ranges from nil (at aspect ratios of 1.2, 1.4, 1.5, and 1.8) to a maximum of 1.85 % at 3.0).

For the results of maximum bending moment coefficient for uniformly distributed load as shown in Tables 2 through 4, the results show satisfactory agreement with the classical solution. The comparison of results (Aginam, 2011) with Finite difference method (FDM) and classical solution for a square plate depicts that the direct variational method has a deviation of about 0.5% with the classical solution while Finite difference method (FDM) has about 8.3% (Table-5).

Also the percentage variation of the Finite difference and Direct variational methods with the classical solution for the mid span moments on the short span for square plate are 6% (for 8 x 8 matrix) and (3 x 3 matrix), respectively. For moment coefficients at the edge of long span, similar comparison shows that the percentage variation for finite difference is 7% and the direct variational method is 2.7% (Table-5).

5.2 Thin rectangular plate with 3 edges simply supported and the other edge clamped

The results of maximum deflection coefficient (α) in isotropic rectangular plate with 3 edges simply supported and other edge clamped are compared with the classical solution (Table-6). For the plate aspect ratios of 1.0 to 2.0 considered, there is satisfactory agreement of the results of direct variational approach with the classical solution. The percentage deviation from the classical solution ranges from nil (at aspect ratios of 1/1.2 to maximum of 2.24 at 1.4%. The validity of the results of the present study is equally amplified when the maximum short span and long span coefficients are compared with the results of the classical solutions. For plate aspect ratios of 0.5 to 2 considered, the percentage variation of results of the present study from the classical solutions for short span moment ranges from nil at plate aspect ratio of 1.10 to maximum value of 2.61% as plate aspect ratio of 0.5 (Table-7). Also, as plate aspect ratio increases, the short span moments increases.

For long span moment coefficient, results converge excellently at 3-term polynomials. Also for plate aspect ratios of 0.5 to 2 considered, there is increase of long span moment coefficient as the plate aspect ratio increases up to 2. Thus, the optimum value of long span coefficients is obtained at the plate aspect ratio of 1.8.

6. CONCLUSION AND CONTRIBUTION TO KNOWLEDGE

In the course of this study, several methods of analyses especially the numerical methods such as finite element, finite difference, finite strip etc were extensively reviewed. The most widely accepted classical solution method, though acknowledged as satisfactory for most Engineering problems, is usually very tedious and rigorous. In view of these antecedent problems, direct variational method under the principle of total potential energy is formulated to circumvent the rigorous procedures inherent in the analysis of classical solution. The study adopted here provides the evaluation of plate analysis without necessarily solving the differential plate equation.

The algebraic functions which must satisfy geometric/essential boundary conditions are carefully constructed. These algebraic equations are then made to satisfy plate equations by minimization principle.
The formulated method is successively applied to:

a) Isotopic thin rectangular plate with two opposite edges simply supported and the remaining opposite edges clamped.
b) Isotopic thin rectangular plate with 3 edges simply supported and the remaining edge clamped.

The loading applied to both plates is uniformly distributed load. The results of the analysis of maximum deflection at the center of span and the maximum positive moments at the center and the maximum negative moments at the supports are found to be in excellent agreement with classical solution. The results equally compares favorably with the numerical methods.

To the best of my knowledge, this is the first attempt to extend the works of direct variational method in the analysis of thin rectangular plates. However, the good agreement of the results of the present study with the classical and numerical methods confirms the validity of the present study. The analytical method which circumvents the rigorous procedures inherent in classical and numerical methods in plate equations is very straightforward, cheap and easy. The method is very handy and could easily be understood by any practising engineer. Therefore, with the knowledge of mathematics, calculus of variation and with programmable calculators, plates of arbitrary boundary conditions can be analyzed for deflection, moments and possibly shear. Thus with maximum size of matrix method in the solution (4x4 in the present case), it makes the use of the present study very attractive. The proposed method has an advantage of having the solution in analytical form which can be used to carry optimization studies. The research method equally enables the determination of deflection, moments and shears at any arbitrary point on the plate unlike numerical methods that give results only at the nodal points.

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REFERENCES

Aginam C. H. (2011) Application of Direct Variational Methods in the Analysis of Thin Rectangular Plates, PhD Thesis submitted to the School of postgraduate Studies, University of Nigeria, Nsukka, Nigeria.

Al-Hassani H. M, Hussain H. M. and Ahmad A.A. (2009) Evaluation of Elastic deflections and Bending Moments for orthotropically Reinforced Slabs supported on three Edges only, *Engineering and Technological Journal,* 27(7).

Dey S. S. (1991) Semi-Numerical Analysis of Rectangular Plates in Bending, *Computers and structures,* 27(7): 315-534.

Gierlinski J. T. and Graves Smith T. R. (1984) The Geometric Non-linear Analysis of Thin-walled Structures by Finite Strips, *Thin-walled Structures,* 2: 27-50.

Hasebe N. and Wang X. (2002) Green Functions for the Bending of thin Plates under various Boundary Conditions and Applications: A Review, *Thin-walled Structures*, 40(7-8): 611-624.

Kaneda K., Iraha S. Takamine T. and Shimabaku K. (2005) An Analytical Study of Rectangular Plates under Triangular Distributed Regional Loads, *Journal of Applied Mechanics,* 7: 17-28.

Mbakogu F. C. and Pavlovic M. N. (2000) Bending of Clamped Orthotropic Rectangular Plates: A Variational Symbolic Approach, *Computers and Structures,* 77(3): 117-128.

Reddy J. N. (2002) Energy Principles and Variational Methods in Applied Mechanics, John Willy and Co, Inc. New Jersey, USA.

Szilard R. (2004) Theories **and** Plate Analysis: Classical, Numerical and Engineering Methods, John Wiley and Co. Inc. New Jersey, USA.

Taylor R. L and Govindjee S. (2004) Solution of Clamped Rectangular Plate Problems, *Communications in Numerical Methods in Engineering*, 20: 757-765.

Ventsel C. and Krauthammer T. (2001) Thin Plates and Shells: Theories, Analysis and Applications, Marcel Dekker, Inc. New York, USA.

Wang C. M. Wang Y. C. and Reddy J. N. (2002) Problems and Remedy for Ritz Method in Determining Stress Resultants of Corner Supported Rectangular Plates, *Computers and Structures,* 80: 145-154.

Zenkour A. M. (2003) Exact Mixed-Classical Solutions for the Bending Analysis of Shear Deformable Rectangular Plates, *Applied Mathematical Modelling,* 27(7): 515-534.