VOL. 7, NO. 10, OCTOBER 2012 ISSN 1819-6608
ARPN Journal of Engineering and Applied Sciences

www.arpnjournals.com

CONTRIBUTION OF SHEAR DEFORMATION IN THE ANALYSIS OF RIGIDLY FIXED PORTAL FRAMES

Okonkwo V. O., Aginam C. H. and Onodagu P. D.
Department of Civil Engineering, Nnamdi Azikiwe University, Awka Anambra State, Nigeria
E-Mail: chukwuraaginam@yahoo.com

ABSTRACT

In this work, the stiffness equations for evaluating the internal stress of rigidly fixed portal frames (considering shear deformation) by the displacement method were generated. But obtaining the equations for the internal stresses required a parametric inversion of the structure stiffness matrix. To circumvent this problem, the flexibility method was used taking advantage of the symmetrical nature of the portal frame and the method of virtual work. These were used to obtain the internal stresses on rigidly fixed portal frames for different cases of external loads when shear deformation is considered. A dimensionless constant α was used to capture the effect of shear deformation in the equations. When it is set to zero, the effect of shear deformation is ignored and the equations become the same as what can be obtained in any structural engineering textbook. These equations were used to investigate the contribution of shear deformation to the calculated internal stresses and how they vary with the ratio of the flexural rigidity of the beam and columns and height to length ratio of the loaded portal frames.

Keywords: portal frames, flexibility method, shear deformation, stiffness matrix.

1. INTRODUCTION

Structural frames are primarily responsible for strength and rigidity of buildings. For simpler single storey structures like warehouses, garages etc portal frames are usually adequate. It is estimated that about 50% of the hot-rolled constructional steel used in the UK is fabricated into single-storey buildings [1]. This shows the increasing importance of this fundamental structural assemblage. The analysis of portal frames are usually done with predetermined equations obtained from structural engineering textbooks or design manuals [2]. It is important to note that most of the equations in these texts were derived with an underlying assumption that deformation of structures due to shearing forces is negligible. This can lead to considerable error in the case of deep beams and in light weight structures were precision is of utmost concern [3]. The twenty first century has seen an astronomical use of computers in the analysis of structures [4] but this has not completely eliminated the use of manual calculations for simple structures and for easy cross-checking of computer output [5]. Hence the need for the development of equations that capture the contribution of shear deformation in portal frames for different loading conditions.

2. DEVELOPMENT AND APPLICATION OF THE MODEL

The analysis of portal frames by the stiffness method requires the determination of the structure's degrees of freedom and the development of the structure's stiffness matrix. For the structure shown in Figure-1(a), the degrees of freedom are as shown in Figure-1(b). I_1 and A_1 are respectively the second moment of inertia and cross-sectional area of the columns while I_2 and A_2 are the second moment of inertia and cross-sectional area of the beam respectively. The stiffness coefficients for the various degrees of freedom considering shear deformation can be obtained from equations developed in Ghali and Neville [6] and Okonkwo [7] and are presented below:

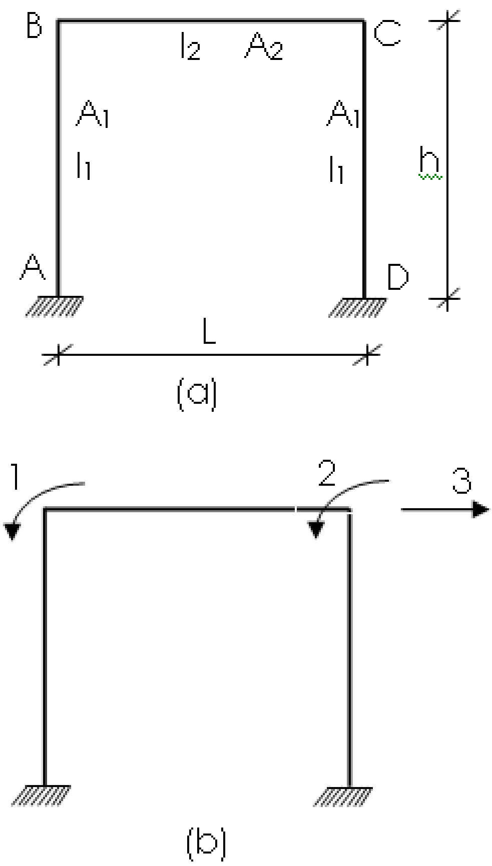


Figure-1. A simple portal frame showing its dimensions and the 3 degrees of freedom.

The structure's stiffness matrix can be written as:

VOL. 7, NO. 10, OCTOBER 2012 ISSN 1819-6608
ARPN Journal of Engineering and Applied Sciences

www.arpnjournals.com

$$K = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \tag{1}$$

Where k_{ij} is the force in coordinate (degree of freedom) i when there is a unit displacement in coordinate (degree of freedom) j. They are as follows:

$$\left.\begin{aligned} k_{11} &= \frac{EI}{l}\left(\frac{\alpha_2+4}{\alpha_2+1}\right)+\frac{EI}{l}\left(\frac{\alpha_1+4}{\alpha_1+1}\right) \\ k_{12} &= \frac{EI}{l}\left(\frac{2-\alpha_2}{\alpha_2+1}\right) \\ k_{13} &= \frac{6EI}{h^2}\left(\frac{1}{\alpha_1+1}\right) \\ k_{22} &= \frac{EI}{l}\left(\frac{\alpha_2+4}{\alpha_2+1}\right)+\frac{EI}{l}\left(\frac{\alpha_1+4}{\alpha_1+1}\right) \\ k_{23} &= \frac{6EI}{h^2}\left(\frac{1}{\alpha_1+1}\right) \end{aligned}\right\} \tag{2}$$

$$k_{33} = \frac{12EI}{h^3}\left(\frac{1}{\alpha_1+1}\right)+\frac{12EI}{h^3}\left(\frac{1}{\alpha_1+1}\right)$$

Where $\alpha_1 = \dfrac{12EI_1}{GA_{1r}h^2}$ (3)

and $\alpha_2 = \dfrac{12EI_2}{GA_{2r}l^2}$ (4)

$A_{1r} = \dfrac{A_1}{\kappa}$ and $A_{2r} = \dfrac{A_2}{\kappa}$, κ is a shape factor which depends on the shape of the member's cross-section. The reduced area A_r of the section can be evaluated from:

$$A_r = \frac{I^2}{\int_A R^2 d\Omega} \tag{5}$$

A_r is the reduced area, I is the second moment of area of the cross section.

$$R = \frac{S}{b} \tag{6}$$

Where *b* is the width of the section, $S = \int_{c_2}^{c_1} y\, da$. c_1 is the distance of the topmost fibre from the neutral axis, c_2 is the distance of the bottom fibre from the neutral axis and *y* is the distance from the neutral axis to any infinitesimal area on the cross section *da* [8].

For rectangular sections κ is 1.2; for a circular cross section it is 1.185 [8] and for a circular tube it is 1/6 [9]. The values of κ for other cross sections are given in Timoshenko and Gere [10].

From Maxwell's Reciprocal theorem and Betti's Law $k_{ij} = k_{ji}$ [11].

When there are external loads on the structure on the structure there is need to calculate the forces in the restrained structure F_o as a result of the external load.

The structure's equilibrium equations are then written as

$$\{F\} = \{F_o\} + [K]\{D\} \tag{7}$$

Jenkins [12]

$$\{F_o\} = \begin{Bmatrix} k_{10} \\ k_{20} \\ k_{30} \end{Bmatrix} \tag{8}$$

Where k_{io} is the force due to external load in coordinate i when the other degrees of freedom are restrained.

$$\{D\} = \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} \tag{9}$$

Where d_i is the displacement in coordinate i.

$$\{F\} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} \tag{10}$$

Where F_i, is the external load with a direction coinciding with the coordinate i.

By making the displacement vector {D} the subject of the formula in equation (7)

$$\{D\} = [K]^{-1}\{F - F_o\} \tag{11}$$

Once {D} is obtained the internal stresses in the frame can be easily obtained by writing the structure's compatibility equations given as:

$$M = M_r + M_1 d_1 + M_2 d_2 + M_3 d_3 \tag{12}$$

Where M is the bending moment at any point on the frame, M_r is the bending moment at the point under consideration in the restrained structure while M_i is the bending moment at that point when there is a unit displacement in coordinate i.

To solve equation (12) there is need to obtain {D}. {D} can be obtained from the inversion of [K] in equation (1). Evaluating the inverse of [K] parametrically (i.e., without substituting the numerical values of E, h, l etc) is a difficult task. This problem is circumvented by using the flexibility method to solve the same problem, taking advantage of the symmetrical nature of the structure and the principle of virtual work.

3. USING THE FLEXIBILITY METHOD

The basic system or primary structure for the structure in Figure-1a is given in Figure-2. The removed redundant forces are depicted with X_1, X_2 and X_3.

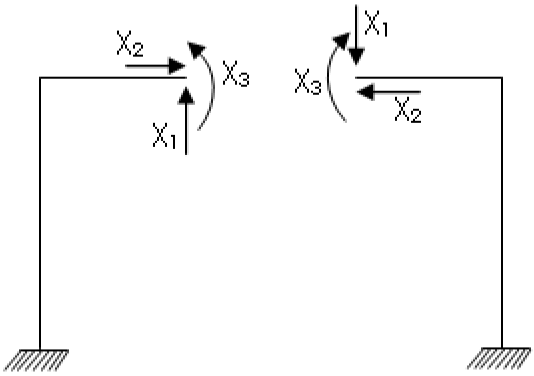


Figure-2. The Basic system showing the removed redundant forces.

The flexibility matrix of the structure can be determined using the principle of virtual work.

By applying the unit load theorem the deflection in beams or frames can be determined for the combined action of the internal stresses, bending moment and shearing forces with

$$D = \int \frac{\bar{M}M}{EI} ds + \int \frac{\bar{V}V}{GA_r} ds \quad (13)$$

$$A_r = \frac{A}{\kappa} \quad (14)$$

Where $\bar{M}$ and $\bar{V}$ are the virtual internal stresses while M and V are the real/actual internal stresses.

E is the modulus of elasticity of the structural material

A is the cross-sectional area of the element

G is the modulus of elasticity in shear, $G = \frac{E}{2(1+v)}$ where v is poisson's ratio κ is as defined earlier [6, 13].

If d_{ij} is the deformation in the direction of i due to a unit load at j then by evaluating equation (13) the following are obtained.

$$d_{11} = \frac{I_1 GA_{2r} l^3 + 6I_2 GA_{2r} l^2 h + 12EI_1 I_2 l}{12EI_1 I_2 GA_{2r}} \quad (14a)$$

$$d_{12} = 0 \quad (14b)$$

$$d_{13} = 0 \quad (14c)$$

$$d_{22} = \frac{2GA_{1r} h^3 + 6EI_1 h}{3EI_1 GA_{1r}} \quad (14d)$$

$$d_{23} = \frac{-h^2}{EI_1} \quad (14e)$$

$$d_{33} = \frac{lI_1 + 2hI_2}{EI_1 I_2} \quad (14f)$$

From Maxwell's Reciprocal theorem and Betti's Law $d_{ij} = d_{ji}$.

The structure's compatibility equations can be written thus:

$$\begin{bmatrix} d_{11} & d_{11} & d_{11} \\ d_{21} & d_{21} & d_{21} \\ d_{31} & d_{31} & d_{31} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} + \begin{Bmatrix} d_{10} \\ d_{20} \\ d_{30} \end{Bmatrix} = 0 \quad (15a)$$

Or alternatively, $fF + d_o = 0$ (15b)

Where F is the vector of redundant forces X_1, X_2, X_3 and d_o is the vector deformation d_{10}, d_{20}, d_{30} due to external load on the basic system (reduced structure).

$$F = f^{-1}(-d_o) \quad (16)$$

However, $f^{-1} = \frac{Adj[f]}{det[f]}$ .. (17) Stroud [14]

$$f^{-1} = \begin{bmatrix} \frac{1}{d_{11}} & 0 & 0 \\ 0 & \frac{d_{33}}{d_{22}d_{33} - d_{23}^2} & \frac{-d_{32}}{d_{22}d_{33} - d_{23}^2} \\ 0 & \frac{-d_{23}}{d_{22}d_{33} - d_{23}^2} & \frac{d_{22}}{d_{22}d_{33} - d_{23}^2} \end{bmatrix} \quad (18)$$

From equations (14a) – (14f)

$$\frac{1}{d_{11}} = \frac{12EI_1 I_2 GA_{2r}}{I_1 GA_{2r} l^3 + 6I_2 GA_{2r} l^2 h + 12EI_1 I_2 l} \quad (19a)$$

$$\frac{d_{33}}{d_{22}d_{33} - d_{23}^2} = \frac{3EI_1 GA_{1r}(lI_1 + 2hI_2)}{2GA_{1r}h^3 lI_1 + 6EI_1^2 hl + GA_{1r}h^4 I_2 + 12EI_1 I_2 h^2} \quad (19b)$$

$$\frac{d_{32}}{d_{22}d_{33} - d_{23}^2} = \frac{-3EI_1 I_2 GA_{1r} h^2}{2GA_{1r}h^3 lI_1 + 6EI_1^2 hl + GA_{1r}h^4 I_2 + 12EI_1 I_2 h^2} \quad (19c)$$

$$\frac{d_{22}}{d_{22}d_{33} - d_{23}^2} = \frac{EI_1 I_2(2GA_{1r}h^3 + 6EI_1 h)}{2GA_{1r}h^3 lI_1 + 6EI_1^2 hl + GA_{1r}h^4 I_2 + 12EI_1 I_2 h^2} \quad (19d)$$

Equation (16) is evaluated to get the redundant forces and these are substituted into the structures force equilibrium (superposition) equations to obtain the bending moments at any point.

$$M = M_o + M_1 X_1 + M_2 X_2 + M_3 X_3 \quad (20)$$

Where M is the required bending moment at a point, M_o is the stress at that point for the reduced structure, M_i is moment at that point when only the redundant force $X_i = 1$ acts on the reduced structure.

For the loaded portal frame of Figure-3, the deformations of the reduced structure due to external loads are:

 ISSN 1819-6608
ARPN Journal of Engineering and Applied Sciences

$$d_{10} = 0 \tag{21a}$$

$$d_{20} = \frac{wh^2l^2}{8EI_1} \tag{21b}$$

$$d_{30} = \frac{-wl^2(lI_1 + 6hI_2)}{24EI_1I_2} \tag{21c}$$

By substituting the values of equations (21a) - (21c) into equations (16)

$$X_1 = 0 \tag{22a}$$

$$X_2 = \frac{-3GA_{1r}(lI_1 + 2hI_2)wh^2l^2}{8(2GA_{1r}h^3lI_1 + 6EI_1^2hl + GA_{1r}h^4I_2 + 12EI_1I_2h^2)} \tag{22b}$$

$$X_3 = \frac{3wh^4l^2GA_{ir}I_2 + 2wh^3l^3GA_{1r}I_1 + 6whl^3EI_1^2 + 36wh^2l^2EI_1I_2}{24(2GA_{1r}h^3lI_1 + 6EI_1^2hl + GA_{1r}h^4I_2 + 12EI_1I_2h^2)} \tag{22c}$$

Evaluating equation (20) for different points on the structure using the force factors obtained in equations (22a) - (22c)

$$M_A = \frac{whl^3I_1(h^2GA_{1r} - 6EI_1)}{12(2GA_{1r}h^3lI_1 + 6EI_1^2hl + GA_{1r}h^4I_2 + 12EI_1I_2h^2)} \tag{23}$$

$$M_B = \frac{-4whl^3I_1(h^2GA_{1r} + 3EI_1)}{24(2GA_{1r}h^3lI_1 + 6EI_1^2hl + GA_{1r}h^4I_2 + 12EI_1I_2h^2)} \tag{24}$$

$$M_C = \frac{-4whl^3I_1(h^2GA_{1r} + 3EI_1)}{24(2GA_{1r}h^3lI_1 + 6EI_1^2hl + GA_{1r}h^4I_2 + 12EI_1I_2h^2)} \tag{25}$$

$$M_D = \frac{whl^3I_1(h^2GA_{1r} - 6EI_1)}{12(2GA_{1r}h^3lI_1 + 6EI_1^2hl + GA_{1r}h^4I_2 + 12EI_1I_2h^2)} \tag{26}$$

For the loaded portal frame of Figure-4, the deformations of the reduced structure due to external loads are

$$d_{10} = \frac{-wl^2(l^2I_1GA_{2r} + 8hlI_2GA_{2r} + 16EI_1I_2)}{128EI_1I_2GA_{2r}} \tag{27a}$$

$$d_{20} = \frac{wh^2l^2}{16EI_1} \tag{27b}$$

$$d_{30} = \frac{-wl^2(lI_1 + 6hI_2)}{24EI_1I_2} \tag{27c}$$

By substituting the values of equations (27a) - (27c) into equations (16):

$$X_1 = \frac{3wl(l^2I_1GA_{2r} + 8hlI_2GA_{2r} + 16EI_1I_2)}{32(I_1GA_{2r}l^2 + 6I_2GA_{2r}lh + 12EI_1I_2)} \tag{28a}$$

$$X_2 = \frac{-2wh^2l^3GA_{1r}}{16(2GA_{1r}h^3lI_1 + 6EI_1^2hl + GA_{1r}h^4I_2 + 12EI_1I_2h^2)} \tag{28b}$$

$$X_3 = \frac{2wl^2(GA_{1r}h^3 + 3EI_1h)(lI_1 + 6hI_2) - 9wl^2h^4I_2GA_{1r}}{48(2GA_{1r}h^3lI_1 + 6EI_1^2hl + GA_{1r}h^4I_2 + 12EI_1I_2h^2)} \tag{28c}$$

Evaluating equation (20) for different points on the structure using the force factors obtained in equations (28a) - (28c)

$$M_A = \frac{whl^2GA_{1r}(8lh^2I_1 + 3h^3I_2) + 6whl^2EI_1(lI_1 + 6hI_2)}{48(2GA_{1r}h^3lI_1 + 6EI_1^2hl + GA_{1r}h^4I_2 + 12EI_1I_2h^2)} - \frac{wl^2(5l^2I_2GA_{2r} + 24lI_2GA_{1r}h + 48EI_1I_2)}{64(I_1GA_{2r}l^2 + 6I_2GA_{2r}lh + 12EI_1I_2)} \tag{29}$$

$$M_B = \frac{2wl^2(GA_{1r}h^3 + 3EI_1h)(lI_1 + 6hI_2) - 9wl^2h^4I_2GA_{1r}}{48(2GA_{1r}h^3lI_1 + 6EI_1^2hl + GA_{1r}h^4I_2 + 12EI_1I_2h^2)} - \frac{wl^2(5l^2I_2GA_{2r} + 24lI_2GA_{1r}h + 48EI_1I_2)}{64(I_1GA_{2r}l^2 + 6I_2GA_{2r}lh + 12EI_1I_2)} \tag{30}$$

$$M_C = \frac{2wl^2(GA_{1r}h^3 + 3EI_1h)(lI_1 + 6hI_2) - 9wl^2h^4I_2GA_{1r}}{48(2GA_{1r}h^3lI_1 + 6EI_1^2hl + GA_{1r}h^4I_2 + 12EI_1I_2h^2)} - \frac{3wl^2(l^2I_1GA_{2r} + 8hlI_2GA_{2r} + 16EI_1I_2)}{64(I_1GA_{2r}l^2 + 6I_2GA_{2r}lh + 12EI_1I_2)} \tag{31}$$

$$M_D = \frac{whl^2GA_{1r}(8lh^2I_1 + 3h^3I_2) + 6whl^2EI_1(lI_1 + 6hI_2)}{48(2GA_{1r}h^3lI_1 + 6EI_1^2hl + GA_{1r}h^4I_2 + 12EI_1I_2h^2)} - \frac{3wl^2(l^2I_1GA_{2r} + 8hlI_2GA_{2r} + 16EI_1I_2)}{64(I_1GA_{2r}l^2 + 6I_2GA_{2r}lh + 12EI_1I_2)} \tag{32}$$

For the loaded portal frame of Figure-5, the deformations of the reduced structure due to external loads are:

$$d_{10} = -\frac{wlh^3}{12EI_1} \tag{33a}$$

$$d_{20} = \frac{wh^2(h^2GA_{1r} + 4EI_1)}{8EI_1GA_{1r}} \tag{33b}$$

$$d_{30} = \frac{-wh^3}{6EI_1} \tag{33c}$$

By substituting the values of equations (33a) - (33c) into equations (16)

$$X_1 = \frac{wh^3I_2GA_{2r}}{I_1GA_{2r}l^2 + 6I_2GA_{2r}lh + 12EI_1I_2} \tag{34a}$$

VOL. 7, NO. 10, OCTOBER 2012 ISSN 1819-6608
ARPN Journal of Engineering and Applied Sciences

www.arpnjournals.com

$$X_2 = \frac{-wh(3lh^2I_1GA_{1r} + 12EI_1^2l + 2h^3I_2GA_{1r} + 24hEI_1I_2)}{8(2GA_{1r}h^2lI_1 + 6EI_1^2l + GA_{1r}h^3I_2 + 12EI_1I_2h)} \quad (34b)$$

$$X_3 = \frac{-wh^5I_2GA_{1r}}{24(2GA_{1r}h^2lI_1 + 6EI_1^2l + GA_{1r}h^3I_2 + 12EI_1I_2h)} \quad (34c)$$

Evaluating equation (18) for different points on the structure using the force factors obtained in equations (34a) - (34c)

$$M_A = -\frac{wh^2(I_1GA_{2r}l^2 + 5GA_{2r}lh + 12EI_1I_2)}{2(I_1GA_{2r}l^2 + 6I_2GA_{2r}lh + 12EI_1I_2)} + \frac{wh^2(9h^2lI_1GA_{1r} + 36lEI_1^2 + 5h^3I_2GA_{1r} + 72hEI_1I_2)}{24(2GA_{1r}h^2lI_1 + 6EI_1^2l + GA_{1r}h^3I_2 + 12EI_1I_2h)} \quad (35)$$

$$M_B = \frac{wh^3I_2GA_{2r}}{2(I_1GA_{2r}l^2 + 6I_2GA_{2r}lh + 12EI_1I_2)} - \frac{wh^5I_2GA_{1r}}{24(2GA_{1r}h^2lI_1 + 6EI_1^2l + GA_{1r}h^3I_2 + 12EI_1I_2h)} \quad (36)$$

$$M_C = -\frac{wh^3I_2GA_{2r}}{2(I_1GA_{2r}l^2 + 6I_2GA_{2r}lh + 12EI_1I_2)} - \frac{wh^5I_2GA_{1r}}{24(2GA_{1r}h^2lI_1 + 6EI_1^2l + GA_{1r}h^3I_2 + 12EI_1I_2h)} \quad (37)$$

$$M_D = -\frac{wh^3I_2GA_{2r}}{2(I_1GA_{2r}l^2 + 6I_2GA_{2r}lh + 12EI_1I_2)} + \frac{wh^2(9h^2lI_1GA_{1r} + 36lEI_1^2 + 5h^3I_2GA_{1r} + 72hEI_1I_2)}{24(2GA_{1r}h^2lI_1 + 6EI_1^2l + GA_{1r}h^3I_2 + 12EI_1I_2h)} \quad (38)$$

For the loaded portal frame of Figure-6, the deformations of the reduced structure due to external loads are:

$$d_{10} = -\frac{Pa^2l}{2EI_1} \quad (39a)$$

$$d_{20} = 0 \quad (39b)$$

$$d_{30} = 0 \quad (39c)$$

By substituting the values of equations (39a) - (39c) into equations (16)

$$X_1 = \frac{6Pa^2I_2GA_{2r}}{I_1GA_{2r}l^2 + 6I_2GA_{2r}lh + 12EI_1I_2} \quad (40a)$$

$$X_2 = 0 \quad (40b)$$

$$X_3 = 0 \quad (40c)$$

Evaluating equation (20) for different points on the structure using the force factors obtained in equations (40a) - (40c)

$$M_A = -Pa\left(1 - \frac{3alI_2GA_{2r}}{I_1GA_{2r}l^2 + 6I_2GA_{2r}lh + 12EI_1I_2}\right) \quad (41)$$

$$M_B = \frac{3Pa^2lI_2GA_{2r}}{I_1GA_{2r}l^2 + 6I_2GA_{2r}lh + 12EI_1I_2} \quad (42)$$

$$M_C = -\frac{3Pa^2lI_2GA_{2r}}{I_1GA_{2r}l^2 + 6I_2GA_{2r}lh + 12EI_1I_2} \quad (43)$$

$$M_D = Pa\left(1 - \frac{3alI_2GA_{2r}}{I_1GA_{2r}l^2 + 6I_2GA_{2r}lh + 12EI_1I_2}\right) \quad (44)$$

For the loaded portal frame of Figure-7, the deformations of the reduced structure due to external loads are:

$$d_{10} = -\frac{Pa^2l}{4EI_1} \quad (45a)$$

$$d_{20} = \frac{Pa^2(3h - a)GA_{1r} + 6EI_1Pa}{6EI_1GA_{1r}} \quad (45b)$$

$$d_{30} = -\frac{Pa^2}{2EI_1} \quad (45c)$$

By substituting the values of equations (45a) - (45c) into equations (16):

$$X_1 = \frac{3Pa^2I_2GA_{2r}}{I_1GA_{2r}l^2 + 6I_2GA_{2r}lh + 12EI_1I_2} \quad (46a)$$

$$X_2 = \frac{-Pa(lI_1 + 2hI_2)[(3h - a)aGA_{1r} + 6EI_1] + 3Pa^2I_2GA_{1r}h^2}{2(2GA_{1r}h^3lI_1 + 6EI_1^2hl + GA_{1r}h^4I_2 + 12EI_1I_2h^2)} \quad (45b)$$

$$X_3 = \frac{-Pa(h - a)(ah^2I_2GA_{1r} + 6hEI_1I_2)}{2(2GA_{1r}h^3lI_1 + 6EI_1^2hl + GA_{1r}h^4I_2 + 12EI_1I_2h^2)} \quad (46c)$$

Evaluating equation (20) for different points on the structure using the force factors obtained in equations (46a) - (46c).

$$M_A = -Pa + \frac{3Pa^2lI_2GA_{2r}}{2(I_1GA_{2r}l^2 + 6I_2GA_{2r}lh + 12EI_1I_2)} + \frac{Pa^2I_2(6EI_1 - GA_{1r}h^2) + Pa(lI_1 + hI_2)[(3h - a)aGA_{1r} + 6EI_1]}{2(2GA_{1r}h^2lI_1 + 6EI_1^2l + GA_{1r}h^3I_2 + 12EI_1I_2h)} \quad (47)$$

$$M_B = \frac{3Pa^2lI_2GA_{2r}}{2(I_1GA_{2r}l^2 + 6I_2GA_{2r}lh + 12EI_1I_2)} - \frac{Pa(h - a)(ahI_2GA_{1r} + 6EI_1I_2)}{2(2GA_{1r}h^2lI_1 + 6EI_1^2l + GA_{1r}h^3I_2 + 12EI_1I_2h)} \quad (48)$$

VOL. 7, NO. 10, OCTOBER 2012 ISSN 1819-6608
ARPN Journal of Engineering and Applied Sciences

www.arpnjournals.com

$$M_C = -\frac{3Pa^2 lI_2 GA_{2r}}{2(I_1 GA_{2r} l^2 + 6I_2 GA_{2r} lh + 12EI_1 I_2)} - \frac{Pa(h-a)(ahI_2 GA_{1r} + 6EI_1 I_2)}{2(2GA_{1r} h^2 lI_1 + 6EI_1^2 l + GA_{1r} h^3 I_2 + 12EI_1 I_2 h)}$$
(49)

$$M_D = -\frac{3Pa^2 lI_2 GA_{2r}}{2(I_1 GA_{2r} l^2 + 6I_2 GA_{2r} lh + 12EI_1 I_2)} + \frac{Pa^2 I_2 (6EI_1 - GA_{1r} h^2) + Pa(lI_1 + hI_2)[(3h-a)aGA_{1r} + 6EI_1]}{2(2GA_{1r} h^2 lI_1 + 6EI_1^2 l + GA_{1r} h^3 I_2 + 12EI_1 I_2 h)}$$
(50)

For the loaded portal frame of Figure-8, the deformations of the reduced structure due to external loads are:

$$d_{10} = -\frac{Pa[aI_1 GA_{2r}(3l-2a) + 6hlI_2 GA_{2r} + 12EI_1 I_2]}{12EI_1 I_2 GA_{2r}} \quad (51a)$$

$$d_{20} = \frac{Pah^2}{2EI_1} \quad (51b)$$

$$d_{30} = -\frac{Pa(aI_1 + 2hI_2)}{2EI_1 I_2} \quad (51c)$$

By substituting the values of equations (51a) – (51c) into equations (16)

$$X_1 = \frac{Pa[aI_1 GA_{2r}(3l-2a) + 6hlI_2 GA_{2r} + 12EI_1 I_2]}{I_1 GA_{2r} l^2 + 6I_2 GA_{2r} lh + 12EI_1 I_2} \quad (52a)$$

$$X_2 = \frac{-3PahI_1 GA_{1r}(l-a)}{2(2GA_{1r} h^2 lI_1 + 6EI_1^2 l + GA_{1r} h^3 I_2 + 12EI_1 I_2 h)} \quad (52b)$$

$$X_3 = \frac{Pa[aI_1(2GA_{1r} h^2 + 6EI_1) + hI_2(GA_{ir} h^2 + 12EI_1)]}{2(2GA_{1r} h^2 lI_1 + 6EI_1^2 l + GA_{1r} h^3 I_2 + 12EI_1 I_2 h)} \quad (52c)$$

Evaluating equation (20) for different points on the structure using the force factors obtained in equations (52a) – (52c)

$M_A =$ . . . (49)

$$-\frac{Pa[GA_{2r}(2l^2 I_1 + 6I_2 lh - 3alI_1 + 2a^2 I_1) + 12EI_1 I_2]}{2(I_1 GA_{2r} l^2 + 6I_2 GA_{2r} lh + 12EI_1 I_2)} + \frac{Pa[GA_{1r} h^3 (3lI_1 - aI_1 + hI_2) + 6EI_1(aI_1 + 2hI_2)]}{2(2GA_{1r} h^2 lI_1 + 6EI_1^2 l + GA_{1r} h^3 I_2 + 12EI_1 I_2 h)}$$
(53)

$M_B =$

$$-\frac{Pa[GA_{2r}(2l^2 I_1 + 6I_2 lh - 3alI_1 + 2a^2 I_1) + 12EI_1 I_2]}{2(I_1 GA_{2r} l^2 + 6I_2 GA_{2r} lh + 12EI_1 I_2)} + \frac{Pa[aI_1(2GA_{1r} h^2 + 6EI_1) + hI_2(GA_{ir} h^2 + 12EI_1)]}{2(2GA_{1r} h^2 lI_1 + 6EI_1^2 l + GA_{1r} h^3 I_2 + 12EI_1 I_2 h)}$$
(54)

$$M_C = -\frac{Pa[aI_1 GA_{2r}(3l-2a) + 6hlI_2 GA_{2r} + 12EI_1 I_2]}{2(I_1 GA_{2r} l^2 + 6I_2 GA_{2r} lh + 12EI_1 I_2)} + \frac{Pa[aI_1(2GA_{1r} h^2 + 6EI_1) + hI_2(GA_{ir} h^2 + 12EI_1)]}{2(2GA_{1r} h^2 lI_1 + 6EI_1^2 l + GA_{1r} h^3 I_2 + 12EI_1 I_2 h)}$$
(55)

$$M_D = -\frac{Pa[aI_1 GA_{2r}(3l-2a) + 6hlI_2 GA_{2r} + 12EI_1 I_2]}{2(I_1 GA_{2r} l^2 + 6I_2 GA_{2r} lh + 12EI_1 I_2)} + \frac{Pa[GA_{1r} h^3 (3lI_1 - aI_1 + hI_2) + 6EI_1(aI_1 + 2hI_2)]}{2(2GA_{1r} h^2 lI_1 + 6EI_1^2 l + GA_{1r} h^3 I_2 + 12EI_1 I_2 h)}$$
(56)

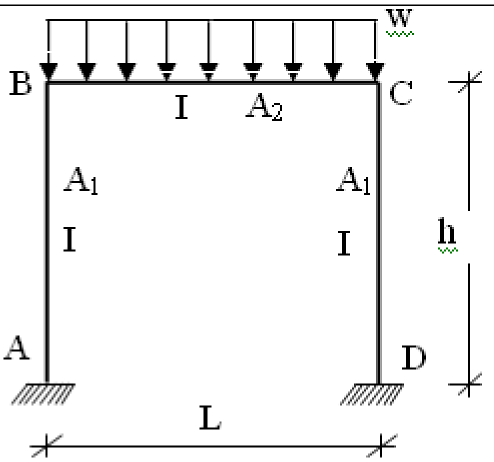


Figure-3.

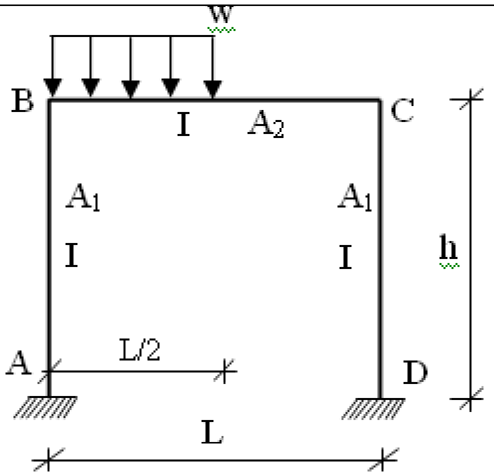


Figure-4.

ARPN Journal of Engineering and Applied Sciences

www.arpnjournals.com

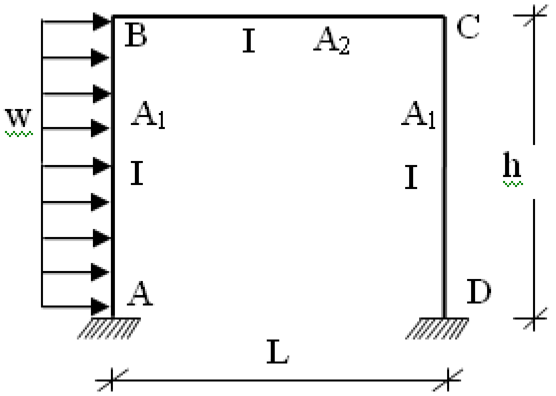


Figure-5.

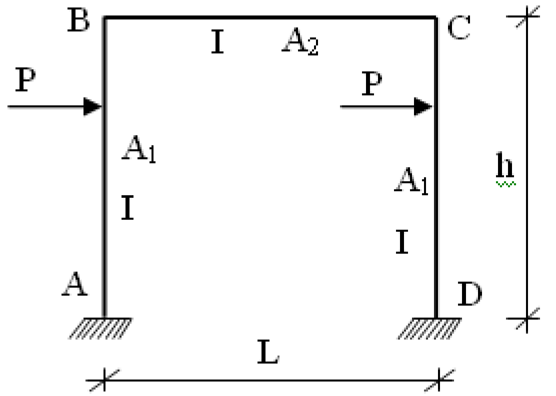


Figure-6.

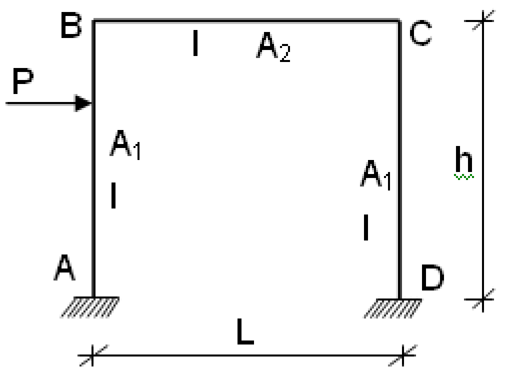


Figure-7.

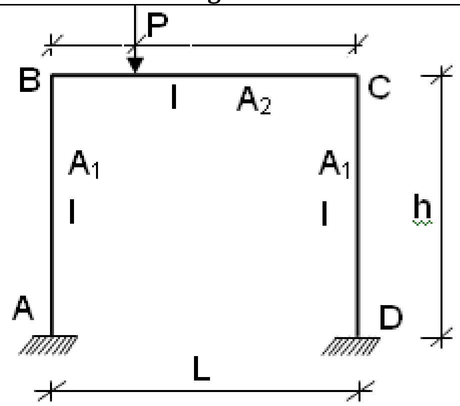
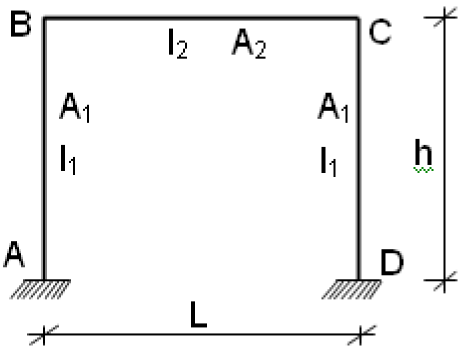


Figure-8.

4. DISCUSSION OF RESULTS

The calculated internal stresses on the loaded frames are summarized in Table-1.

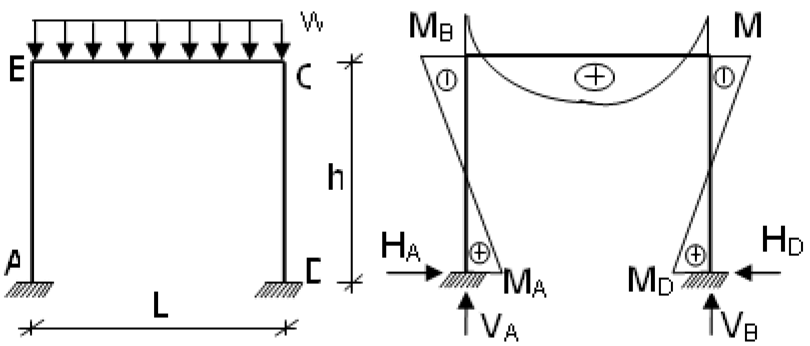
Table-1. Internal stresses on rigidly fixed loaded portal frames.



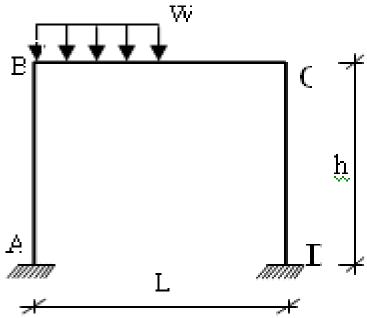
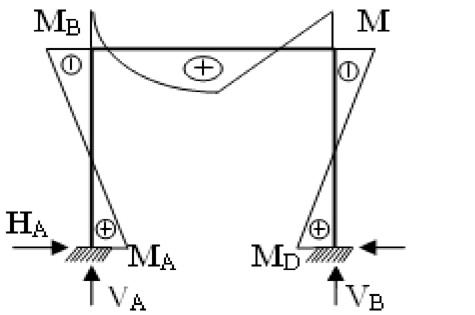
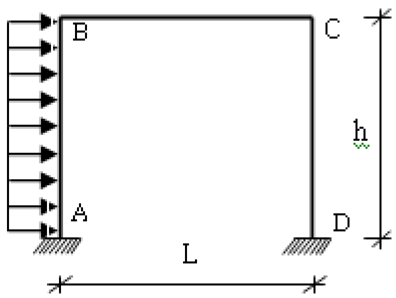
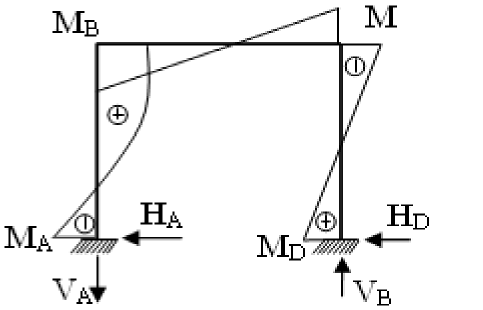
A_1 = Cross-sectional area of the columns
I_1 = Second moment of area of the column cross-section
A_2 = Cross-sectional area of the beam
I_2 = Second moment of area of the beam cross-section

$$\alpha_1 = \frac{12EI_1}{h^2 GA_{1r}} \qquad \alpha_2 = \frac{12EI_2}{l^2 GA_{2r}}$$

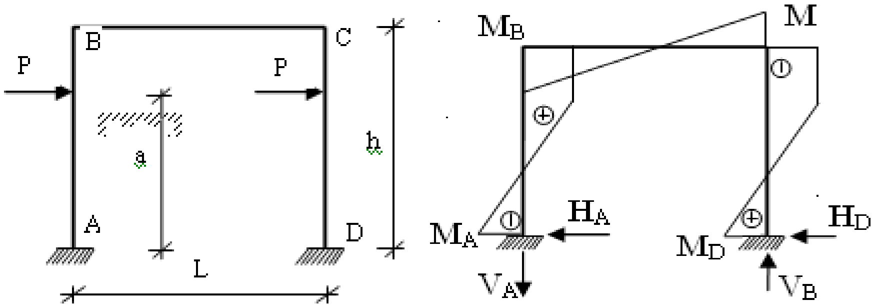
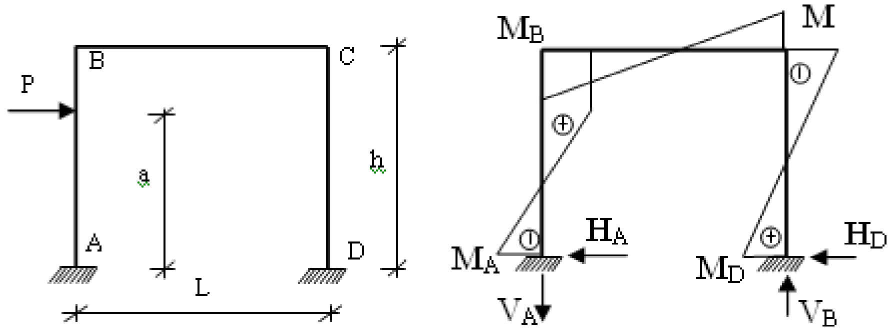
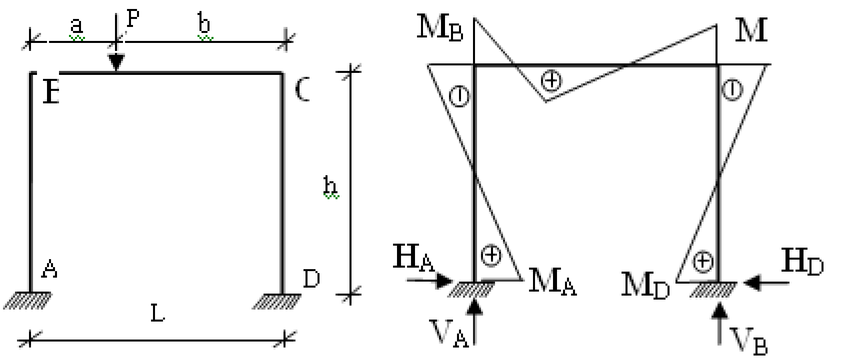
$$\beta = \frac{\alpha_2}{\alpha_1}$$

S/No.	Loaded frame	Remarks
1		See equations (23) - (26)

ISSN 1819-6608
ARPN Journal of Engineering and Applied Sciences

	$M_A = M_D = \frac{wl^3 I_1(2-\alpha_1)}{12[2(2lI_1 + hI_2)+\alpha_1\ (lI_1 + 2hI_2)]}$ $M_B = M_C = \frac{-wl^3 I_1(4+\alpha_1)}{12[2(2lI_1 + hI_2)+\alpha_1\ (lI_1 + 2hI_2)]}$ $V_A = V_D = \frac{wl}{2}$ $H_A = H_D = \frac{wl^3 I_1}{lI_1(4+\alpha_1) + hI_2(2+\alpha_1)}$	
2	 $M_A = \frac{wl^2[2(8lI_1 + 3hI_2)+\alpha_1\ (lI_1 + 6hI_2)]}{48[2(2lI_1 + hI_2)+\alpha_1\ (lI_1 + 2hI_2)]} - \frac{wl^3 I_1(5h + 24\beta l + 4\alpha_2 h)}{64h(lI_1 + 6hI_2+\alpha_2\ lI_1)}$ $M_B = \frac{wl^2[(4+\alpha_1)(lI_1 + 6hI_2) - 18hI_2]}{48[2(2lI_1 + hI_2)+\alpha_1\ (lI_1 + 2hI_2)]} - \frac{wl^3 I_1(5h + 24\beta l + 4\alpha_2 h)}{64h(lI_1 + 6hI_2+\alpha_2\ lI_1)}$ $M_C = \frac{wl^2[(4+\alpha_1)(lI_1 + 6hI_2) - 18hI_2]}{48[2(2lI_1 + hI_2)+\alpha_1\ (lI_1 + 2hI_2)]} - \frac{wl^2(3lI_1I_2 + 24hI_2^2 + 4\ \alpha_2\ lI_1I_2)}{64(lI_1I_2 + 6hI_2^2+\alpha_2\ lI_1I_2)}$ $M_D = \frac{wl^2[2(8lI_1 + 3hI_2)+\alpha_1\ (lI_1 + 6hI_2)]}{48[2(2lI_1 + hI_2)+\alpha_1\ (lI_1 + 2hI_2)]} - \frac{wl^2(3lI_1I_2 + 24hI_2^2 + 4\ \alpha_2\ lI_1I_2)}{64(lI_1I_2 + 6hI_2^2+\alpha_2\ lI_1I_2)}$ $V_D = \frac{wl(3lI_1 + 24hI_2 + 4\ \alpha_2\ lI_1)}{32(lI_1 + 6hI_2+\alpha_2\ lI_1)}$ $V_A = \frac{wl}{2} - V_D$ $H_A = H_D = \frac{wl^3 I_1}{4h(lI_1(4+\alpha_1) + 2hI_2(1+\alpha_1))}$	See equations (29) - (32)
3	 $M_A = -\frac{wh^2(I_1 l + 5hI_2+\alpha_2\ lI_1)}{2(I_1 l + 6hI_2+\alpha_2\ lI_1)} + \frac{wh^2(9lI_1 + 36\ \alpha_1\ lI_1 + 5hI_2 + 72\ \alpha_1\ hI_2)}{12(4lI_1+\alpha_1\ lI_1 + 2hI_2 + 2\ \alpha_1\ hI_2)}$ $M_B = \frac{wh^3 I_2}{2(I_1 l + hI_2+\alpha_2\ lI_1)} - \frac{wh^3 I_2}{12(4lI_1+\alpha_1\ lI_1 + 2hI_2 + 2\ \alpha_1\ hI_2)}$ $M_C = -\frac{wh^3 I_2}{2(I_1 l + hI_2+\alpha_2\ lI_1)} - \frac{wh^3 I_2}{12(4lI_1+\alpha_1\ lI_1 + 2hI_2 + 2\ \alpha_1\ hI_2)}$ $M_D = -\frac{wh^3 I_2}{2(I_1 l + hI_2+\alpha_2\ lI_1)} + \frac{wh^2(9lI_1 + 36\ \alpha_1\ lI_1 + 5hI_2 + 72\ \alpha_1\ hI_2)}{12(4lI_1+\alpha_1\ lI_1 + 2hI_2 + 2\ \alpha_1\ hI_2)}$ $V_A = V_B = \frac{wh^3 I_2}{I_1 l^2 + 6I_2 lh+\alpha_2\ I_1 l^2}$ $H_D = \frac{wh(3lI_1+\alpha_1\ lI_1 + 2hI_2 + 24\ \alpha_1\ hI_2)}{4(4lI_1+\alpha_1\ lI_1 + 2hI_2 + 2\ \alpha_1\ hI_2)}$ $H_A = wh - H_D$	See equations (35) - (38)

ARPN Journal of Engineering and Applied Sciences

4	 $M_A = -Pa + \frac{3Pa^2 I_2}{lI_1 + 6hI_2 + \alpha_2 lI_1}$ $M_B = \frac{3Pa^2 I_2}{lI_1 + 6hI_2 + \alpha_2 lI_1}$ $M_C = -\frac{3Pa^2 I_2}{lI_1 + 6hI_2 + \alpha_2 lI_1}$ $M_D = Pa - \frac{3Pa^2 I_2}{lI_1 + 6hI_2 + \alpha_2 lI_1}$ $V_A = V_B = \frac{6Pa^2 I_2}{l^2 I_1 + 6lhI_2 + \alpha_2\, l^2 I_1}$ $H_A = H_B = P$	
5	 $M_A = -Pa + \frac{3Pa^2 I_2}{2(lI_1 + 6hI_2 + \alpha_2 lI_1)} + \frac{Pa^2 h^2 I_2(\alpha_1 - 2) + Pa(lI_1 + hI_2)[2a(3h - a) + \alpha_1\, h^2]}{2h^2(4lI_1 + \alpha_1\, lI_1 + 2hI_2 + 2\,\alpha_1\, hI_2)}$ $M_B = \frac{3Pa^2 I_2}{2(lI_1 + 6hI_2 + \alpha_2 lI_1)} - \frac{Pa(h - a)(2aI_2 + \alpha_1\, hI_2)}{2h(4lI_1 + \alpha_1\, lI_1 + 2hI_2 + 2\,\alpha_1\, hI_2)}$ $M_C = -\frac{3Pa^2 I_2}{2(lI_1 + 6hI_2 + \alpha_2 lI_1)} - \frac{Pa(h - a)(2aI_2 + \alpha_1\, hI_2)}{2h(4lI_1 + \alpha_1\, lI_1 + 2hI_2 + 2\,\alpha_1\, hI_2)}$ $M_D = -\frac{3Pa^2 I_2}{2(lI_1 + 6hI_2 + \alpha_2 lI_1)} + \frac{Pa^2 h^2 I_2(\alpha_1 - 2) + Pa(lI_1 + hI_2)[2a(3h - a) + \alpha_1\, h^2]}{2h^2(4lI_1 + \alpha_1\, lI_1 + 2hI_2 + 2\,\alpha_1\, hI_2)}$ $V_A = V_B = \frac{3Pa^2 I_2}{l^2 I_1 + 6lhI_2 + \alpha_2\, l^2 I_1}$ $H_D = \frac{Pa(lI_1 + 2hI_2)[2a(3h - a) + \alpha_1\, h^2] - 3Pa^2 h^2 I_2}{2h(4h^2 lI_1 + \alpha_1\, h^2 lI_1 + 2h^3 I_2 + 2\,\alpha_1\, h^3 I_2)}$ $H_A = P - H_D$	See equations (47) - (50)
6	 $M_A = \frac{Pa[2(3lI_1 - aI_1 + hI_2) + \alpha_1\,(aI_1 + 2hI_2)]}{2(4lI_1 + \alpha_1\, lI_1 + 2hI_2 + 2\,\alpha_1\, hI_2)} - \frac{Pa[2l^2 I_1 + 6lhI_2 - 3alI_1 + 2a^2 I_1 + \alpha_1\, l^2 I_1]}{2l(lI_1 + 6hI_2 + \alpha_2 lI_1)}$ $M_B = -\frac{Pa[2l^2 I_1 + 6lhI_2 - 3alI_1 + 2a^2 I_1 + \alpha_1\, l^2 I_1]}{2l(lI_1 + 6hI_2 + \alpha_2 lI_1)} + \frac{Pa[aI_1(4 + \alpha_1) + 2hI_2(1 + \alpha_1)]}{2(4lI_1 + \alpha_1\, lI_1 + 2hI_2 + 2\,\alpha_1\, hI_2)}$	See equations (53) - (56)

VOL. 7, NO. 10, OCTOBER 2012 ISSN 1819-6608
ARPN Journal of Engineering and Applied Sciences

$$M_C = -\frac{Pa[aI_1(3l-2a)+6hlI_2+\alpha_2\, l^2 I_1]}{2(lI_1+6hI_2+\alpha_2 lI_1)} + \frac{Pa[aI_1(4+\alpha_1)+2hI_2(1+\alpha_1)]}{2(4lI_1+\alpha_1\, lI_1+2hI_2+2\,\alpha_1\, hI_2)}$$

$$M_D = \frac{Pa[2(3lI_1-aI_1+hI_2)+\alpha_1\,(aI_1+2hI_2)]}{2(4lI_1+\alpha_1\, lI_1+2hI_2+2\,\alpha_1\, hI_2)} - \frac{Pa[aI_1(3l-2a)+6hlI_2+\alpha_2\, l^2 I_1]}{2(lI_1+6hI_2+\alpha_2 lI_1)}$$

$$V_D = \frac{Pa[aI_1(3l-2a)+6hlI_2+\alpha_2\, l^2 I_1]}{l^2(lI_1+6hI_2+\alpha_2\, lI_1)} \quad V_A = P - V_D$$

$$H_A = H_B = \frac{3PaI_1(l-a)}{h(4lI_1+\alpha_1\, lI_1+2hI_2+2\,\alpha_1\, hI_2)}$$

The effect of shear deformation is captured by the dimensionless constant α and is taken as the ratio of the end translational stiffness to the shear stiffness of a member.

$$\alpha_1 = \frac{12EI_1}{h^3}\cdot\frac{h}{GA_{1r}} = \frac{12EI_1}{h^2 GA_{1r}} \qquad (57)$$

$$\alpha_2 = \frac{12EI_2}{l^3}\cdot\frac{l}{GA_{2r}} = \frac{12EI_2}{l^2 GA_{2r}} \qquad (58)$$

$$\beta = \frac{\alpha_2}{\alpha_1} \qquad (59)$$

When the shear deformation in the columns is ignored, $\alpha_1 = 0$, and likewise when shear deformation in the beam is ignored $\alpha_2 = 0$. If shear deformation is ignored in the whole structure, $\alpha_1 = \alpha_2 = 0$.

The internal stress equations enable an easy investigation into the contribution of shear deformation to the internal stresses of statically loaded frames for different kinds of external loads.

For frame 1 (Figure-3), the moment at A, M_A is given by equation (23). The contribution of shear deformation in the column ΔM_A is given by:

$$\Delta M_A = M_{A(from\ equation\ 21)} - M_{A(from\ equation\ 21\ when\ \alpha_1=0)}$$

$$= \frac{wl^3 I_1}{12}\left[\frac{2-\alpha_1}{lI_1(4+\alpha_1)+2hI_2(1+\alpha_1)} - \frac{1}{2lI_1+hI_2}\right] \qquad (60)$$

Equation (60) gives the contribution of shear deformation to M_A as a function of h, l, I_1, I_2 and α.

For a concrete portal frame of length l = 5m, height h = 4m and the columns' depth about the axis of bending H = 0.4m; equation (60) was evaluated to show how shear deformation varied with I_1/I_2. The result was plotted in Figure-9. From Figure-9 it would be observed that at as $I_1/I_2 => 0$ the value of $\Delta M_A => 0$. This implies that an increasing flexural rigidity EI_2 of the beam relative to that of the column reduces the contribution of shear deformation to the value of the internal stresses. As the value of $I_1/I_2 => \infty$ the value of $\Delta M_A => -0.0223$, this is the maximum shear contribution per unit load w. By varying the height to length ratio h/l of the portal frame at a constant I_1/I_2 of 10, 0000, the peak ΔM_A is observed at $h/l = 0$ to be -3.125 (See Figure-10). This also represents a 10.4117% decrease in the bending moment value obtained when shear deformation was ignored.

In like manner by evaluating the shear contribution in the beam, ΔM_B for varying I_1/I_2 of the portal frame, Figure-11 is obtained. From Figure-11 it would be observed that at as $I_1/I_2 => 0$ and as $I_1/I_2 => \infty$ the value of $\Delta M_B => 0$, however a maximum ΔM_B is observed at $I_1/I_2 = 0.4043$ to be 0.0111. By varying the height to length ratio h/l of the portal frame at a constant I_1/I_2 of 0.4043, the peak ΔM_B is observed at $h/l = 0.059$ to be -0.20216 (See Figure-12). This also represents a 19.407% decrease in the bending moment value obtained when shear deformation was ignored.

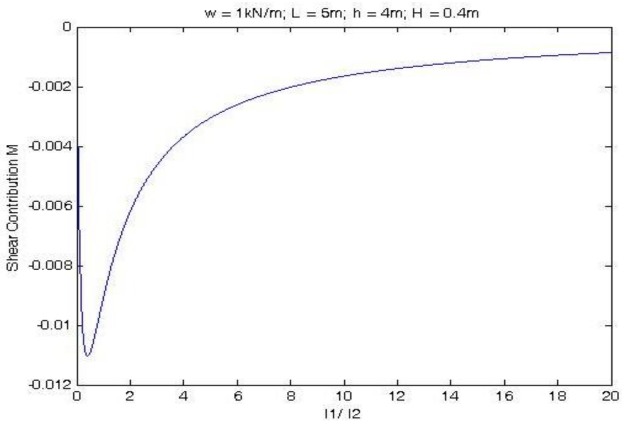


Figure-9. A graph of ΔM_A against I_1/I_2.

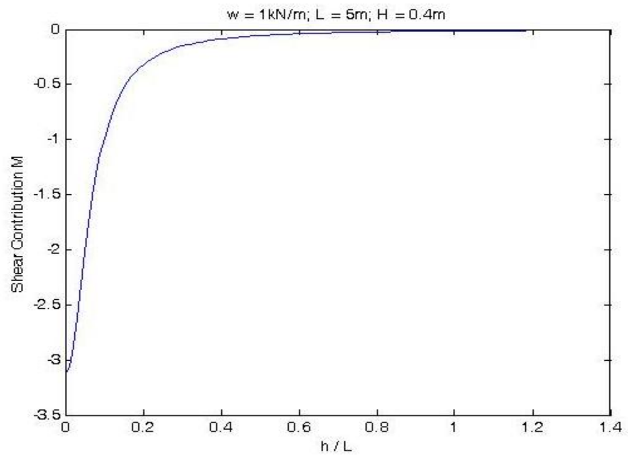


Figure-10. A graph of ΔM_A against h/L.

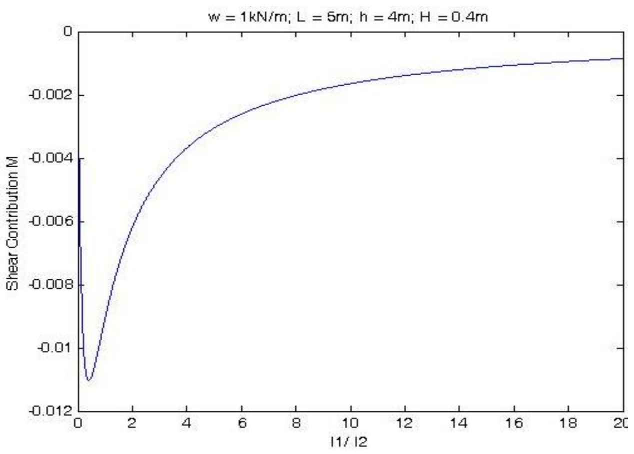


Figure-11. A graph of ΔM_B against I_1/I_2.

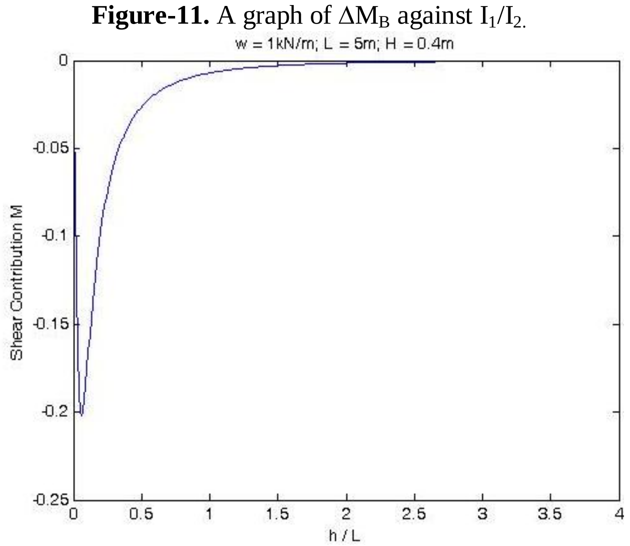


Figure-12. A graph of ΔM_B against h/L.

5. CONCLUSIONS

The use of the flexibility method has simplified the analysis and a summary of the results are presented in Table-1. The equations in Table-1 would enable an easy evaluation of the internal stresses in loaded rigidly fixed portal frames considering the effect of shear deformation.

From a detailed analysis of frame 1 (Figure-3), it was observed that though the contribution of shear deformation is generally very small, it can be significant under certain conditions. The contribution of shear deformation ΔM_A was maximum when I_1/I_2 was very large and the ratio h/l very small. ΔM_B Was also significant when I_1/I_2 is in the neighbourhood of 0.4043 and h/l around 0.059. In the analysis of the portal frame of Figure-3 the negligence of shear deformation cannot be justified when the existing circumstances is close to the ones outlined above. This detailed analysis can be extended to the other loaded frames (Figures 4-8) using the equations in Table-1. These would enable the determination of safe conditions for ignoring shear deformation under different kinds of loading. However Table-1 offers simplified formula for evaluating the bending moment in rigidly fixed portal frames (considering shear deformation) under different kinds of loading.

REFERENCES

[1] Graham R and Alan P. 2007. Single Storey Buildings: Steel Designer's Manual. 6th Edition. Blackwell Science Ltd, United Kingdom.

[2] Reynolds C. E. and Steedman J. C. 2001. Reinforced Concrete Designer's Handbook. 10th Edition. E and FN Spon, Taylor and Francis Group, London, U.K.

[3] Moy S. S. J. 1996. Plastic Methods for Steel and Concrete Structures. 2nd Edition. Macmillan Press Ltd, London, U.K.

[4] Samuelsson A. and Zienkiewi cz O. C. 2006. Review: History of the Stiffness Method. International Journal for Numerical Methods in Engineering. 67: 149-157.

[5] Hibbeler R. C. 2006. Structural Analysis. 6th Edition. Pearson Prentice Hall, New Jersey, USA.

[6] Ghali A and Neville A. M. 1996. Structural Analysis: A Unified Classical and Matrix Approach. 3rd Edition. Chapman and Hall London, U.K.

[7] Okonkwo V. O. 2012. Computer-aided Analysis of Multi-storey Steel Frames. M. Eng. Thesis, Nnamdi Azikiwe University, Awka, Nigeria.

[8] Vitor D. S. 2006. Mechanics and Strength of Material, Springer-Verlag, Heidelberg New York, USA.

[9] Renton J. D. 2002. Elastic Beams and Frames. 2nd Edition. Horwood Publishing Limited West Sussex. England.

[10] Timoshenko S. P. and Gere J. M. 1972. Mechanics of Materials. Van Nostrand New York, USA.

[11] Leet K. M. and Uang C. 2002. Fundamental of Structural Analysis. McGraw-Hill, New York, USA.

[12] Jenkins W. M. 1990. Structural Analysis using computers. 1st edition. Longman Group Limited, Hong Kong.

[13] Nash W. 1998. Schaum's Outline of Theory and Problems of Strength of Materials. 4th Edition. McGraw-Hill Companies, New York, USA.

[14] Stroud K. A. 1995. Engineering Mathematics. 4th Edition. Macmillan Press Ltd, London, U.K.