



UNSTEADY FLOW OF A FLUID PARTICLE SUSPENSION BETWEEN TWO PARALLEL PLATES SUDDENLY SET IN MOTION WITH SAME SPEED

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ABSTRACT

In this paper we consider a fluid particle suspension filling the region between two rigid parallel plates, assuming that initially the plates as well as the fluid are at rest. The unsteady flow generated by moving suddenly the two plates in their own plane is studied. Taking the Laplace Transform of the equations of motion with the appropriate central and boundary conditions, the Laplace Transforms of the fluid velocity, dust particle velocity, volume flux of the fluid and volume flux of the dust particles across a plane normal to the flow per unit width and the skin friction are obtained. Numerical Inversion is carried out using the procedure of Honig and Hirdes [1] and their variation is studied with respect to diverse fluid parameters, space variable and time. The variation of the flow field quantities is studied numerically and the results are presented through graphs.

Keywords: fluid particle suspension, unsteady flow, Laplace transform, numerical inversion, dust particle velocity.

INTRODUCTION

The study of dynamics of fluid particle suspension has applications in many branches of Engineering as well as Environmental, Physical and Biological branches. The flow of dissolved micro molecules of fibre suspensions in paper making, the using dust in gas cooling systems to enhance heat transfer processes, the movement of dust laid in air and the flow of blood through Arteries are some examples of fluid particle suspensions. Saffman [2] proposed a model to describe the flow of a fluid particle suspension which on account of its relative simplicity has attracted the attention of many workers in the field of fluid dynamics. In this model the effect of the particle dust suspended in the fluid is described by two parameters; the concentration of the dust particles and a relaxation time τ_0 which measures the rate at which the velocity of a dust particle adjusts to changes in the fluid velocity and depends upon the size of the individual particles. He derived the equations of motion for this fluid under the following simplifying assumption:

- (i) The dust particles are of uniform size and uniform shape;
- (ii) The fluid velocity, the dust particle velocity and number density are respectively denoted by the vector field $\vec{u}_s(\vec{x}, t)$ and scalar field $N(\vec{x}, t)$.
- (iii) The bulk concentration of the dust (mass of dust particles in a volume element/mass of the fluid volume element containing the dust particles) is very small so that the net effect of the dust on the fluid is equivalent to an extra force $\kappa N(\vec{u} - \vec{u}_s)$ per unit volume (κ is a constant, the Reynolds number of the relative motion of dust and the fluid is small compared with unity so that the force between dust and fluid particle is proportional to the relative velocity).

Assuming the fluid to be incompressible and the bulk concentration is small; Saffman [2] has derived the equations of motion of a dusty fluid (or a fluid particle suspension) as:

$$\text{div } \vec{q} = 0 \quad (1)$$

$$\rho \left(\frac{\partial \vec{q}}{\partial t} + (\vec{q} \cdot \nabla) \vec{q} \right) = -\nabla p + \mu \nabla^2 \vec{q} - K N(\vec{q} - \vec{q}_s) \quad (2)$$

$$m \left(\frac{\partial \vec{q}_s}{\partial t} + (\vec{q} \cdot \nabla) \vec{q}_s \right) = -K(\vec{q}_s - \vec{q}) \quad (3)$$

$$\text{div } \vec{q}_s = 0 \quad (4)$$

where

P = Pressure (less than hydrostatic pressure)

ρ = density

μ = viscosity of the fluid

when there is no suspension of dust particles, assuming the dust particles are spheres of radius a , $K = 6\pi\mu U a$ using Stokes drag formula.

In this paper we study the flow of a fluid particle suspension between two horizontal rigid parallel plates suddenly set in motion with the same speed say U . Initially the plates as well as fluid in between are assumed to be at rest. Suddenly at time $t = 0$ the plates start moving in their own plane with velocity U , in the same direction. The study of this problem is motivated by a remarkable paper of Erdogan [3] in which he studied some unsteady unidirectional viscous fluid flows generated by an impulsive motion of a boundary on a sudden application of



a pressure gradient. One of the problems he attempted in is the unsteady flow generated by the sudden movement of the two plates with the same constant velocity U . He has obtained the solution in a series form and estimated the flow field variables for diverse values of space variables and time. In this paper, we consider the problem for the case of a fluid particle suspension. We take the Laplace Transform for the governing equations with relevant boundary and initial conditions and obtain the analytical expressions for the flow field quantities in Laplace Transforms domain. In view of the complexity of these expressions, analytical inversion is not possible and hence a numerical approach is adopted to find the quantities in space time domain by inverting the expression in Laplace transform domain to the space time domain by inverting the expressions in Laplace transform domain to the space time domain. The variation of flow field variables is studied numerically.

MATHEMATICAL FORMULATION OF THE PROBLEM

Suppose a fluid particle suspension whose flow field equations are given through equations (1) to (4), be bounded by two rigid boundaries at $y = -h$ and $y = h$. Let these as well as the fluid be at rest initially. Let us assume that the fluid starts suddenly due to the motion with the same speed U of the upper and lower plates.

In view of the physics of the problem, we assume that the fluid velocity and dust particle velocity are in the form $\vec{q} = (u(y, t), 0, 0)$ and $\vec{q}^* = (u^*(y, t), 0, 0)$

Assuming N to be constant, the equations of continuity for the velocities \vec{q} and \vec{q}^* are satisfied and we see that the velocity components $u(y, t)$ and $u^*(y, t)$ are governed by:

$$\rho \frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial y^2} + K N (u - u^*) \quad (5)$$

$$m \frac{\partial u^*}{\partial t} = -K (u^* - u) \quad (6)$$

These are to be solved with the initial condition

$$u(y, 0) = 0 \text{ for } -h \leq y \leq h \quad (7)$$

and the boundary conditions

$$u(\pm h, t) = U \quad (8)$$

We notice that the fluid velocity u satisfies the no slip condition on the boundary. The dust phase velocity may not satisfy the no slip conditions because the condition of slip or no slip of dust on the boundary depends very much upon the initial condition imposed on the dust phase. The particles have a tendency to be within

the core region and as such we do not impose any condition on the dust particle velocity on the boundary.

In the present problem, as time t goes to ∞ , the fluid velocity $u(y, t)$ and dust particle velocity $u^*(y, t)$ have to approach U . This is due to the fact that the fluid moves eventually at the same speed as that of the plates. Hence we can write the velocity:

$$u(y, t) = U (1 - f(y, t)) \quad (9)$$

$$u^*(y, t) = U (1 - f^*(y, t)) \quad (10)$$

where $f(y, t)$ and $f^*(y, t)$ are non dimensional functions. Let us introduce the non dimensionalisation scheme

$$u = U \tilde{u}, y = h \tilde{y}, t = \frac{\rho}{\mu} \tilde{t}, u^* = U \tilde{u}^* \quad (11)$$

where $\tilde{u}, \tilde{y}, \tilde{t}, \tilde{u}^*$ are non dimensional.

Using the non dimensionalisation scheme in the equations and dropping the tildes, we have

$$R \frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial y^2} + R_1 (f^* - f) \quad (12)$$

$$R_2 \frac{\partial f^*}{\partial t} = (f - f^*) \quad (13)$$

where we have taken

$$u(y, t) = (1 - f(y, t)) \quad (14)$$

$$u^*(y, t) = (1 - f^*(y, t)) \quad (14)$$

where

$$R = \frac{\rho U h}{\mu}, R_1 = \frac{k_1 N h^2}{\mu}, R_2 = \frac{\tau U}{R} \quad (15)$$

Using equations (9) and (10) along with the initial and boundary conditions (7) and (8), we get

$$f(y, 0) = 1, f^*(y, 0) = 1 \text{ and } f(\pm 1, t) = 0 \forall t \quad (16)$$

We obtain f and f^* by solving equations (12) and (13) using the conditions in (16).

SOLUTION OF THE PROBLEM

As we are dealing with an initial value problem, we shall make use of technique of Laplace transform and try to find the variables in the Laplace transform domain. Taking Laplace transform of equations (12) and (13) with respect to the time variable and using the notation;

$$\bar{F}(y, s) = L \{F(y, t)\} = \int_{t=0}^{\infty} e^{-st} F(y, t) dt \quad (17)$$

after considerable Algebra and decoupling the equations for f and f^* , we get



$$\frac{d^2 \bar{f}}{ds^2} - \alpha^2 \bar{f} = -\frac{\alpha^2}{s} \tag{18}$$

$$\text{where } \alpha^2 = \left[R + \frac{sR_1 R_2}{1+sR_2} \right] \tag{19}$$

and

$$\bar{f}^* = \frac{\bar{f} + R_2}{(1+sR_2)} \tag{20}$$

Boundary conditions in (16) give rise to

$$\bar{f}(\pm 1, s) = 0 \tag{21}$$

Solving the equation (18) using the conditions (21), we notice that

$$\bar{f}(y, s) = \frac{1}{s} - \frac{1}{s} \frac{\cosh \alpha y}{\cosh \alpha} \tag{22}$$

Using equation (22) in equation (20), we have

$$f^*(y, s) = \frac{1}{s} - \frac{1}{s} \frac{\cosh \alpha y}{(1+sR_2) \cosh \alpha} \tag{23}$$

Taking Laplace transform of $u(y, t)$ and $u^*(y, t)$ given in (14), we get

$$\bar{u}(y, s) = \frac{1}{s} - \bar{f}(y, s) = \frac{1}{s} - \frac{1}{s} \frac{\cosh \alpha y}{\cosh \alpha} \tag{24}$$

$$\bar{u}^*(y, s) = \frac{1}{s} - f^*(y, s) = \frac{1}{s} \frac{\cosh \alpha y}{(1+sR_2) \cosh \alpha} \tag{25}$$

By taking the inverse Laplace transform of (24) and (25), we get the non dimensional fluid velocity $u(y, t)$ and non dimensional dust particle velocity $u^*(y, t)$.

VOLUME FLUX OF THE FLUID AND DUST

The volume flux Q across a plane normal to the flow and per unit width of the plane is given by

$$Q = \int_{-1}^1 u(y, t) dy \tag{26}$$

And the volume flux Q^* of the dust particles across a plane normal to the flow and per unit width of the plane is given by

$$Q^* = \int_{-1}^1 u^*(y, t) dy \tag{27}$$

These in Laplace transform domain are respectively seen to be

$$\bar{Q}(s) = 2 \left[\frac{1}{s} - \frac{1}{s} \frac{\cosh \alpha}{\cosh \alpha} \right] \tag{28}$$

$$\bar{Q}^*(s) = \frac{2}{s} \frac{\cosh \alpha}{(1+sR_2) \cosh \alpha} \tag{29}$$

SKIN FRICTION ON THE PLATE Y = 1

The frictional force per unit area exerted by the fluid on the plate $y = 1$ in dimensional form is given by:

$$\bar{\tau}_{yx} = \mu \frac{\partial u}{\partial y}$$

After the necessary non dimensionalisation the non dimensional skin friction on the plate $y = 1$ is given by:

$$\bar{\tau}_{yx} = \left(\frac{\partial u}{\partial y} \right) \text{ at } y = 1$$

and this its Laplace transform is given by:

$$\bar{\tau}_{yx} = \left(\frac{\partial \bar{u}}{\partial y} \right) \tag{30}$$

Thus we have the Laplace transforms of the flow field variables given by:

$$\bar{u}(y, s), \bar{u}^*(y, s), \bar{Q}(s), \bar{Q}^*(s) \text{ and } \bar{\tau}_{yx}(s)$$

These quantities depend on α and s where α is a function of s . In view of this, the derivation of analytical expressions of the inverse Laplace transforms of these in space time domain or in time domain is a herculian task. Hence the nature of the problem demands the use of a numerical inversion procedure.

NUMERICAL WORK AND DISCUSSIONS

The expressions of $\bar{u}(y, s), \bar{u}^*(y, s), \bar{Q}(s), \bar{Q}^*(s)$ and $\bar{\tau}_{yx}(s)$ are inverted making use of numerical inversion procedure of Laplace transform proposed by Honig and Hirdes [1] given in appendix. These are calculated for given values of r, t, y and t .

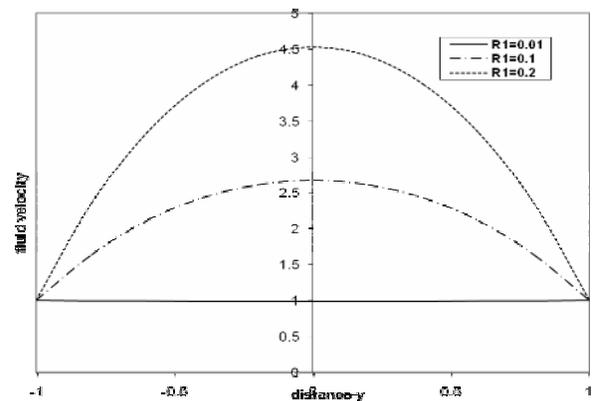


Figure-1. Variation of fluid velocity with different values of R1.

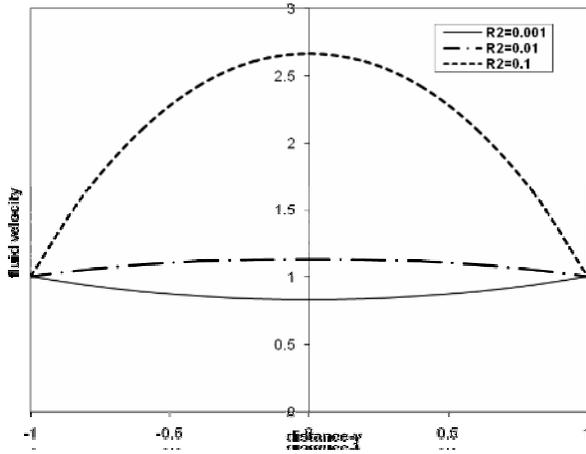


Figure-2. Variation of fluid velocity with different values of R2.

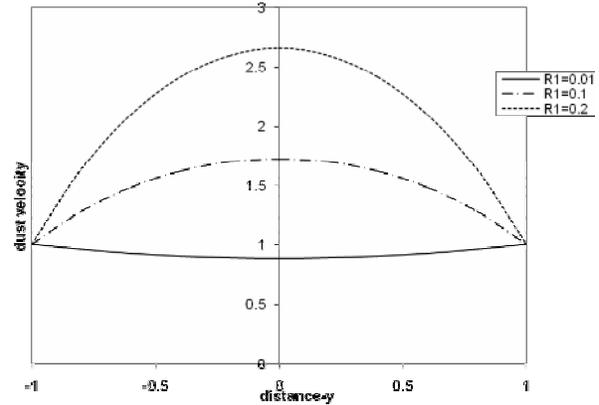


Figure-5. Variation of fluid Velocity with different values of R1.

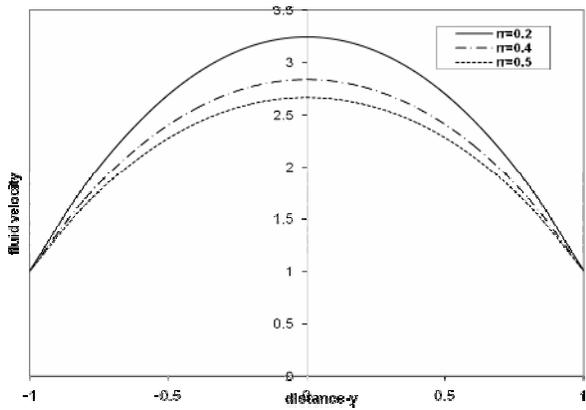


Figure-3. Variation of fluid velocity with different values of R.

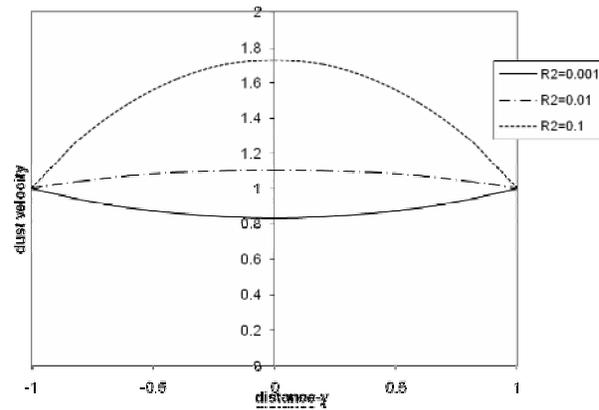


Figure-6. Variation of dust velocity with different values of R2.

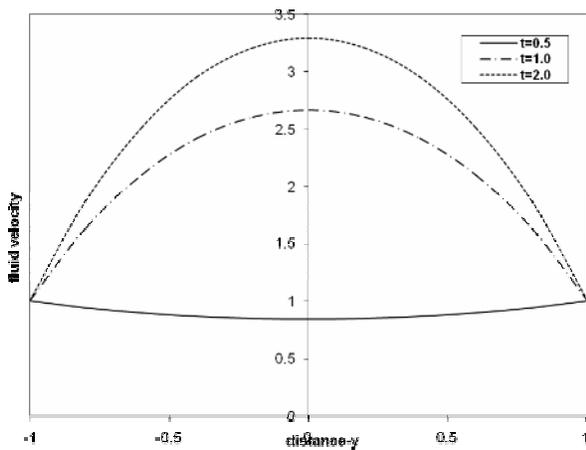


Figure-4. Variation of fluid velocity with different values of t.

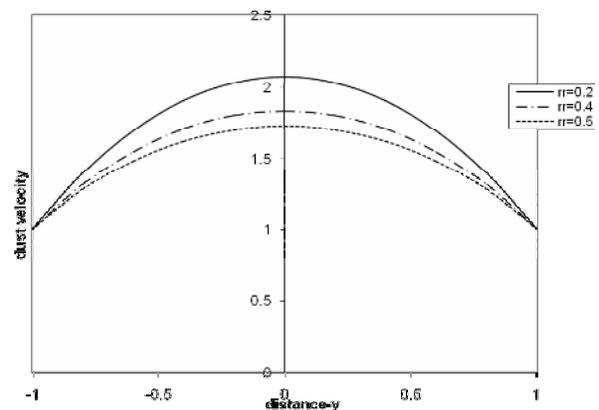


Figure-7. Variation of dust velocity with different values of R.

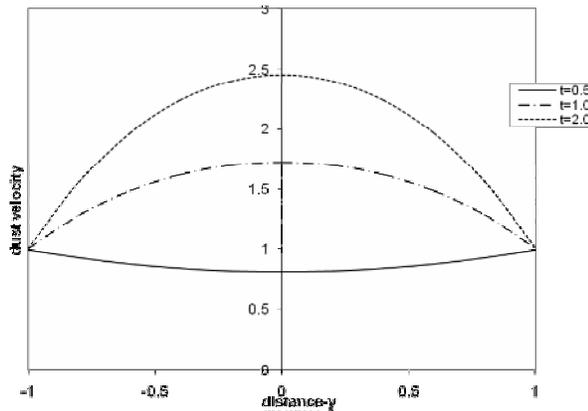


Figure-8. Variation of dust velocity with different values of t .

In Figure-1 and Figure-5, we plot fluid velocity and dust velocity for $R = 0.5$, $R_2 = 0.1$, $t = 1$, for different values of R_1 . As R_1 increases, the fluid velocity $\bar{u}(y, s)$ and the dust velocity $\bar{u}^*(y, s)$ increases at any y .

Figure-2 and Figure-6, are graphs for the fluid velocity and dust velocity for $R = 0.5$, $R_1 = 0.1$, $t = 1$, for different values of R_2 . As R_2 increases, the fluid velocity $\bar{u}(y, s)$ and dust velocity $\bar{u}^*(y, s)$ increasing at any y . Figure-3 and Figure-7, shows the graphs for the fluid velocity and dust velocity for $R_1 = 0.1$, $R_2 = 0.1$, $t = 1$, for different values of R . The fluid velocity $\bar{u}(y, s)$ and dust velocity $\bar{u}^*(y, s)$ decreasing at any y as R increases.

Figure-4 and Figure-8, are the graphs for the fluid velocity and dust velocity for $R = 0.5$, $R_1 = 0.1$, $R_2 = 0.1$, for different values of t . The fluid velocity $\bar{u}(y, s)$ and the dust velocity $\bar{u}^*(y, s)$ increases at any y as t increases.

In all the cases, the dust velocity is less than the fluid velocity at any y .

REFERENCES

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APPENDIX

Numerical inversion technique of Laplace transform due to Honig and Hirdes

In all the problems, as has been indicated, the analytical expressions for the flow variables are obtained in the Laplace transform domain in terms of (y, s) and we have to invert these into (y, t) domain. Analytical expression of these inverse Laplace transforms seems to be out of reach as functions $\bar{f}(y, s)$ under consideration are complicated. In view of this, it is necessary to adopt a suitable numerical inversion technique. We have inverted the relevant functions $\bar{f}(y, s)$ making use of a standard numerical inversion procedure proposed by Honig and Hirdes [1] and this procedure is explained below for a quick reference.

Let $\bar{f}(s)$ be the Laplace transform of a given function $f(t)$. The inversion formula of the Laplace transform states that:

$$f(t) = \frac{1}{2\pi i} \int_{c^*-i\infty}^{c^*+i\infty} e^{st} \bar{f}(s) ds \quad (1)$$

where c^* is an arbitrary constant greater than all the real parts of the singularities of $f(t)$.

Taking $s = c^* + iy$, we get

$$f(t) = \frac{e^{c^*t}}{2\pi} \int_{-\infty}^{\infty} e^{iyt} \bar{f}(c^* + iy) dy \quad (2)$$

This integral can be approximated by:

$$f(t) = \frac{e^{c^*t}}{2\pi} \sum_{k=-\infty}^{\infty} e^{ikt\Delta y} \bar{f}(c^* + ik\Delta y) \Delta y \quad (3)$$

Taking $\Delta y = \pi/t_1$, we get

$$f(t) = \frac{e^{c^*t}}{t_1} \operatorname{Re} \left[\frac{1}{2} \bar{f}(c^*) + \sum_{k=1}^{\infty} e^{ik\pi t/t_1} \bar{f}(c^* + ik\pi/t_1) \right] \quad (4)$$

For numerical purposes this is approximated by the function

$$f_N(t) = \frac{e^{c^*t}}{t_1} \operatorname{Re} \left[\frac{1}{2} \bar{f}(c^*) + \sum_{k=1}^N e^{ik\pi t/t_1} \bar{f}(c^* + ik\pi/t_1) \right] \quad (5)$$

where N is a sufficiently large integer chosen such that $e^{c^*t} \operatorname{Re} \left[e^{iN\pi t/t_1} \bar{f}(c^* + iN\pi/t_1) \right] < \varepsilon$ and ε is a prescribed small positive number that corresponds to the degree of accuracy to be achieved. Formula above is the numerical inversion formula valid for $2t_1 \geq t \geq 0$ [1].

In particular, we choose $t = t_1$, and obtain

$$f_N(t) = \frac{e^{c^*t}}{t} \operatorname{Re} \left[\frac{1}{2} \bar{f}(c^*) + \sum_{k=1}^N (-1)^k \bar{f}(c^* + ik\pi/t) \right] \quad (6)$$