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THE CONVERGENCE AND ORDER OF THE 3-POINT BLOCK EXTENDED BACKWARD DIFFERENTIATION FORMULA

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ABSTRACT

In this paper, we consider the fully implicit 3-point Block Extended Backward Differentiation Formula for solving stiff initial value problems. The iterative block method is proven to be convergent by establishing zero stability and consistency conditions. Numerical results are given to show the effect of zero stability and consistency. The accuracy is seen to improve as the step length tends to zero. The order of the method is also shown to be 6.

Keywords: convergence, order of block method, blocks extended backward differentiation formula.

INTRODUCTION

Consider the first order stiff initial value problem (IVP)

$$y' = f(x,y) \quad y(a) = y_0 \quad x \in [a,b] \tag{1}$$

Such differential equations occur in many fields of engineering science and in particular, they appear in electrical circuit, vibrations, chemical reactions, kinetics etc.

Developing methods for solving (1) still remains a challenge in modern numerical analysis. Sequential methods among them include (Curtiss *et al*., 1952; Hall *et al*., 1985; Dahlquist, 1963; Cash, 1980; Suleiman *et al*., 1989). Block methods for solving (1) can be found in (Fatunla, 1991; Ibrahim *et al*., 2007; Musa *et al*., 2011; Nasir *et al*., 2011; Musa *et al*., 2012). The convergence of block methods for solving (1) using block backward differentiation formula (BBDF) has been studied in (Ibrahim *et al*., 2011). The block extended backward differentiation formula (BEBDF) that approximates the solution of (1) is proposed in (Musa *et al*., 2012) and has the general form:

$$\sum_{j=0}^{5} \alpha_{j,i} y_{n+j-2} = h\beta_{k,i}, f_{n+k} + h\beta_{k+1}, f_{n+k+1}, \quad k = i = 1,2,3. \tag{2}$$

It was developed in quest for higher order A-stable block methods for stiff IVPs. The method improves the accuracy and order of the BBDF method. An extra future point y_{n+4} is involved, which is predicted using conventional backward differentiation formula. The method also approximates the solution at 3-point simultaneously and it is A-stable. For i=1, 2 and 3, it is given by:

$$\begin{aligned}
y_{n+1} &= -\frac{1}{80}y_{n-2} + \frac{1}{8}y_{n-1} - \frac{3}{4}y_n + \frac{25}{16}y_{n+2} + \frac{3}{40}y_{n+3} - \frac{3}{2}hf_{n+1} - \frac{3}{4}hf_{n+2} \\
y_{n+2} &= -\frac{3}{25}y_{n-2} + y_{n-1} - 4y_n + 12y_{n+1} - \frac{197}{25}y_{n+3} + 12hf_{n+2} + \frac{12}{5}hf_{n+3} \\
y_{n+3} &= \frac{394}{14919}y_{n-2} - \frac{2925}{14919}y_{n-1} + \frac{9600}{14919}y_n - \frac{18700}{14919}y_{n+1} + \frac{26550}{14919}y_{n+2} \\
&\quad + \frac{8820}{14919}hf_{n+3} - \frac{600}{14919}hf_{n+4}
\end{aligned} \tag{3}$$

respectively. More details on the method can be found in (Musa *et al*., 2012).

An acceptable linear multistep method (LMM) must be convergent. Consistency and zero stability are the necessary and sufficient conditions for convergence of a LMM. According to (Lambert, 1973), consistency controls the magnitude of the local truncation error while zero stability controls the manner in which the error is propagated at each step of the calculation. A method which is not both consistent and zero stable is rejected outright and has no practical interest. This paper proves the convergence of the method (3) by establishing zero stability and consistency conditions. The order of the method will also be determined.

ORDER OF THE METHOD

The following definitions given in (Lambert, 1973) will be used to establish the order of the method (3).

Definition

The general linear multistep method (LMM) is defined by:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h\sum_{j=0}^{k} \beta_j f_{n+j} \tag{4}$$

where α_j and β_j are constants, $\alpha_k \neq 0$, α_0 and β_0 cannot be zero at the same time.

Definition

The order of the LMM (4) and its associated linear operator given by:

$$L[y(x);h]=\sum_{j=0}^{k}[\alpha_j y(x+jh)-h\beta_j y'(x+jh)] \qquad (5)$$

is defined as a unique integer p such that $C_q=0, q=0(1)p$, and $C_{p+1}\neq 0$, where the C_q are constants defined by:

$$\begin{aligned}
C_0 &= \alpha_0+\alpha_1+\alpha_2+\ldots+\alpha_k\\
C_1 &= \alpha_1+2\alpha_2+\ldots+k\alpha_k-(\beta_0+\beta_1+\beta_2+\ldots+\beta_k)\\
C_q &= \frac{1}{q!}(\alpha_1+2^q\alpha_2+\ldots+k^q\alpha_k)\\
&\quad -\frac{1}{(q-1)!}(\beta_1+2^{q-1}\beta_2+\ldots+k^{q-1}\beta_k),\\
&\quad q=2,3,\ldots,k
\end{aligned} \qquad (6)$$

We extend the above definitions to the method (3) as follows:

Definition

The method (3) can be defined in general matrix form as:

$$\sum_{j=0}^{1}A_j^{*}Y_{m-j}=h\sum_{j=0}^{2}B_{j-1}^{*}F_{m+j-1} \qquad (7)$$

where A_0^*, A_1^*, B_{-1} B_0^* and B_1^* are square matrices defined by:

$$A_0^*=\begin{pmatrix}1 & -\frac{25}{16} & -\frac{3}{40}\\ -12 & 1 & \frac{197}{25}\\ \frac{18700}{14919} & -\frac{26550}{14919} & 1\end{pmatrix},\ A_1^*=\begin{pmatrix}\frac{1}{80} & -\frac{1}{8} & \frac{3}{4}\\ \frac{3}{25} & -1 & 4\\ -\frac{394}{14919} & \frac{2925}{14919} & -\frac{9600}{14919}\end{pmatrix},$$

$$B_{-1}^*=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix},\ B_0^*=\begin{pmatrix}-\frac{3}{2} & -\frac{3}{4} & 0\\ 0 & 12 & \frac{12}{5}\\ 0 & 0 & \frac{8820}{14919}\end{pmatrix},\ B_1^*=\begin{pmatrix}0&0&0\\0&0&0\\-\frac{600}{14919}&0&0\end{pmatrix},$$

and Y_m, Y_{m-1}, F_{m-1}, F_m, F_{m+1} are column vectors defined by:

$$Y_m=\begin{pmatrix}y_{n+1}\\y_{n+2}\\y_{n+3}\end{pmatrix},\ Y_{m-1}=\begin{pmatrix}y_{n-2}\\y_{n-1}\\y_n\end{pmatrix},\ F_{m-1}=\begin{pmatrix}f_{n-2}\\f_{n-1}\\f_n\end{pmatrix},\ F_m=\begin{pmatrix}f_{n+1}\\f_{n+2}\\f_{n+3}\end{pmatrix},$$

$$F_{m+1}=\begin{pmatrix}f_{n+4}\\f_{n+5}\\f_{n+6}\end{pmatrix}.$$

Equation (7) can be re-written as:

$$\begin{pmatrix}\frac{1}{80} & -\frac{1}{8} & \frac{3}{4}\\ \frac{3}{25} & -1 & 4\\ -\frac{394}{14919} & \frac{2925}{14919} & -\frac{9600}{14919}\end{pmatrix}\begin{pmatrix}y_{n-2}\\y_{n-1}\\y_n\end{pmatrix}+\begin{pmatrix}1 & -\frac{25}{16} & -\frac{3}{40}\\ -12 & 1 & \frac{197}{25}\\ \frac{18700}{14919} & -\frac{26550}{14919} & 1\end{pmatrix}\begin{pmatrix}y_{n+1}\\y_{n+2}\\y_{n+3}\end{pmatrix}$$

$$=h\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}f_{n-2}\\f_{n-1}\\f_n\end{pmatrix}+h\begin{pmatrix}-\frac{3}{2} & -\frac{3}{4} & 0\\ 0 & 12 & \frac{12}{5}\\ 0 & 0 & \frac{8820}{14919}\end{pmatrix}\begin{pmatrix}f_{n+1}\\f_{n+2}\\f_{n+3}\end{pmatrix}+h\begin{pmatrix}0&0&0\\0&0&0\\-\frac{600}{14919}&0&0\end{pmatrix}\begin{pmatrix}f_{n+4}\\f_{n+5}\\f_{n+6}\end{pmatrix} \qquad (8)$$

Let A_0^*, A_1^*, B_{-1}^*, B_0^*, and B_1^* be block matrices defined by

$A_0^*=\begin{pmatrix}A_3 & A_4 & A_5\end{pmatrix}$, $A_1^*=\begin{pmatrix}A_0 & A_1 & A_2\end{pmatrix}$, $B_{-1}^*=\begin{pmatrix}B_0 & B_1 & B_2\end{pmatrix}$, $B_0^*=\begin{pmatrix}B_3 & B_4 & B_5\end{pmatrix}$, and $B_1^*=\begin{pmatrix}B_6 & B_7 & B_8\end{pmatrix}$.

where

$$A_0=\begin{pmatrix}\frac{1}{80}\\ \frac{3}{25}\\ -\frac{394}{14919}\end{pmatrix},\ A_1=\begin{pmatrix}-\frac{1}{8}\\ -1\\ \frac{2925}{14919}\end{pmatrix},\ A_2=\begin{pmatrix}\frac{3}{4}\\ 4\\ -\frac{9600}{14919}\end{pmatrix},$$

$$A_3=\begin{pmatrix}1\\ -12\\ \frac{18700}{14919}\end{pmatrix},\ A_4=\begin{pmatrix}-\frac{25}{16}\\ 1\\ -\frac{26550}{14919}\end{pmatrix},\ A_5=\begin{pmatrix}-\frac{3}{40}\\ \frac{197}{25}\\ 1\end{pmatrix},$$

$$B_0=\begin{pmatrix}0\\0\\0\end{pmatrix},\ B_1=\begin{pmatrix}0\\0\\0\end{pmatrix},\ B_2=\begin{pmatrix}0\\0\\0\end{pmatrix},\ B_3=\begin{pmatrix}-\frac{3}{2}\\0\\0\end{pmatrix},\ B_4=\begin{pmatrix}-\frac{3}{4}\\12\\0\end{pmatrix},$$

$$B_5=\begin{pmatrix}0\\ \frac{12}{5}\\ \frac{8820}{14919}\end{pmatrix},\ B_6=\begin{pmatrix}0\\0\\-\frac{600}{14919}\end{pmatrix},\ B_7=\begin{pmatrix}0\\0\\0\end{pmatrix},\ B_8=\begin{pmatrix}0\\0\\0\end{pmatrix}.$$

Definition

The order of the block method (7) and its associated linear operator given by:

$$L\left[y(x);h\right]=\sum_{j=0}^{k=5}\left[A_j y(x+jh)\right]-h\sum_{j=0}^{k+1}\left[B_j y'(x+jh)\right] \qquad (9)$$

is a unique integer p such that $C_q = 0, q = 0(1)p$ and $C_{p+1} \neq 0$; where the C_q are constant column) matrices defined by:

$$C_0 = A_0 + A_1 + A_2 + ... + A_k$$
$$C_1 = A_1 + 2A_2 + ... + kA_k - (\beta_0 + B_1 + B_2 + ... + B_{k+1})$$
$$C_q = \frac{1}{q!}(A_1 + 2^q A_2 + ... + k^q A_k) - \frac{1}{(q-1)!}(B_1 + 2^{q-1} B_2 + ... + (k+1)^{q-1} B_{k+1}) \quad (10)$$

For $q = 0(1)6$, we have

$$C_0 = A_0 + A_1 + A_2 + A_3 + A_4 + A_5 = 0$$
$$C_1 = \left(A_1 + 2.A_2 + 3.A_3 + 4.A_4 + 5.A_5\right) - \left(B_0 + B_1 + B_2 + B_3 + B_4 + B_5 + B_6\right) = 0$$
$$C_2 = \frac{1}{2!}\left(A_1 + 2^2.A_2 + 3^2.A_3 + 4^2.A_4 + 5^2.A_5\right) - \frac{1}{1!}\left(B_1 + 2^1.B_2 + 3^1.B_3 + 4^1.B_4 + 5^1.B_5 + 6^1.B_6\right) = 0$$
$$C_3 = \frac{1}{3!}\left(A_1 + 2^3.A_2 + 3^3.A_3 + 4^3.A_4 + 5^3.A_5\right) - \frac{1}{2!}\left(B_1 + 2^2.B_2 + 3^2.B_3 + 4^2.B_4 + 5^2.B_5 + 6^2.B_6\right) = 0$$
$$C_4 = \frac{1}{4!}\left(A_1 + 2^4.A_2 + 3^4.A_3 + 4^4.A_4 + 5^4.A_5\right) - \frac{1}{3!}\left(B_1 + 2^3.B_2 + 3^3.B_3 + 4^3.B_4 + 5^3.B_5 + 6^3.B_6\right) = 0$$
$$C_5 = \frac{1}{5!}\left(A_1 + 2^5.A_2 + 3^5.A_3 + 4^5.A_4 + 5^5.A_5\right) - \frac{1}{4!}\left(B_1 + 2^4.B_2 + 3^4.B_3 + 4^4.B_4 + 5^4.B_5 + 6^4.B_6\right) = 0$$
$$C_6 = \frac{1}{6!}\left(A_1 + 2^6.A_2 + 3^6.A_3 + 4^6.A_4 + 5^6.A_5\right) - \frac{1}{5!}\left(B_1 + 2^5.B_2 + 3^5.B_3 + 4^5.B_4 + 5^5.B_5 + 6^5.B_6\right) = 0$$
$$C_7 = \frac{1}{7!}\left(A_1 + 2^7.A_2 + 3^7.A_3 + 4^7.A_4 + 5^7.A_5\right) - \frac{1}{6!}\left(B_1 + 2^6.B_2 + 3^6.B_3 + 4^6.B_4 + 5^6.B_5 + 6^6.B_6\right) = \begin{pmatrix} \frac{-1}{280} \\ \frac{-2}{35} \\ -\frac{690}{34811} \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ -0 \end{pmatrix} \quad (11)$$

Therefore the formula (3) is of order 6, with error constant

$$\begin{pmatrix} \frac{-1}{280} \\ \frac{-2}{35} \\ -\frac{690}{34811} \end{pmatrix}.$$

CONVERGENCE OF THE METHOD

Convergence is an essential property that every acceptable linear multistep method must possess. This section proves the convergence of the method (3). According to (Lambert, 1973), consistency and zero stability are the necessary conditions for the convergence of any numerical method. We shall therefore begin with the following theorem and definitions (as given in Lambert, 1973) which relate to the general LMM:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} \quad (12)$$

and then establish new definitions that relate to the fully implicit 3-point BEBDF method. A proof of consistency and zero stability of the method will then follow.

Theorem

The necessary and sufficient conditions for the linear multistep method (12) to be convergent are that it is consistent and zero stable.

Details of the prove can be found in (Henrici, 1962).

Definition

A LMM is said to be consistent if its order $p \geq 1$.

Therefore from (6), it follows that the LMM (12) is consistent if and only if the following conditions are satisfied:

$$\sum_{j=0}^{k} \alpha_j = 0$$
$$\sum_{j=0}^{k} j\alpha_j = \sum_{j=0}^{k} \beta_j = 0 \quad (13)$$

See (Lambert 1973)

Definition

The LMM (12) is said to be zero stable if no root of the first characteristic polynomial has modulus greater than one; and if every root with modulus one is simple. See (Lambert, 1973).

Building on this, we now extend the above theorem and definitions to the BEBDF method as follows:

Theorem

The necessary and sufficient conditions for the BEBDF method (7) to be convergent are that it is consistent and zero stable.

Proof

It suffices to show that (7) is consistent and zero stable. These are shown in subsections 3.1 and 3.2.

Definition

The BEBDF is said to be consistent if its order $p \geq 1$. Therefore from (10), it follows that the BEBDF method (3) is consistent if and only if the following conditions are satisfied:

$$\sum_{j=0}^{5} A_j = 0$$
$$\sum_{j=0}^{5} jA_j = \sum_{j=0}^{6} B_j = 0 \tag{14}$$

where A_j and B_j are as previously defined.

Definition

The BEBDF method (3) is said to be zero stable if no root of the first characteristic polynomial has modulus greater than one, and that with modulus one is simple.

Consistency of the BEBDF method

In this subsection, it is shown that the BEBDF satisfies the consistency conditions given in definition 3.5. From what followed in section 2, it can be concluded that the order of the BEBDF method is >1.

Let $A_0, A_1, ..., A_5$ be as previously defined. Then

$$\sum_{j=0}^{5} A_j = A_0 + A_1 + A_2 + A_3 + A_4 + A_5$$
$$= \begin{pmatrix} \frac{1}{80} \\ \frac{3}{25} \\ -\frac{394}{14919} \end{pmatrix} + \begin{pmatrix} -\frac{1}{8} \\ -1 \\ \frac{2925}{14919} \end{pmatrix} + \begin{pmatrix} \frac{3}{4} \\ 4 \\ -\frac{9600}{14919} \end{pmatrix}$$
$$+ \begin{pmatrix} 1 \\ -12 \\ \frac{18700}{14919} \end{pmatrix} + \begin{pmatrix} -\frac{25}{16} \\ 1 \\ -\frac{26550}{14919} \end{pmatrix} + \begin{pmatrix} -\frac{3}{40} \\ \frac{197}{25} \\ 1 \end{pmatrix} \tag{15}$$
$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Hence the first condition in (14) is satisfied.

$$\sum_{j=0}^{5} jA_j = 0.A_0 + 1.A_1 + 2.A_2 + 3.A_3 + 4.A_4 + 5.A_5$$
$$= 0.\begin{pmatrix} \frac{1}{80} \\ \frac{3}{25} \\ -\frac{394}{14919} \end{pmatrix} + 1.\begin{pmatrix} -\frac{1}{8} \\ -1 \\ \frac{2925}{14919} \end{pmatrix} + 2.\begin{pmatrix} \frac{3}{4} \\ 4 \\ -\frac{9600}{14919} \end{pmatrix}$$
$$+ 3.\begin{pmatrix} 1 \\ -12 \\ \frac{18700}{14919} \end{pmatrix} + 4.\begin{pmatrix} -\frac{25}{16} \\ 1 \\ -\frac{26550}{14919} \end{pmatrix} + 5.\begin{pmatrix} -\frac{3}{40} \\ \frac{197}{25} \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{-9}{4} \\ \frac{72}{5} \\ \frac{2740}{4973} \end{pmatrix} \tag{16}$$

$$\sum_{j=0}^{6} B_j = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -\frac{3}{2} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -\frac{3}{4} \\ 12 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{12}{5} \\ \frac{8820}{14919} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -\frac{600}{14919} \end{pmatrix} \tag{17}$$
$$= \begin{pmatrix} -\frac{9}{4} \\ \frac{72}{5} \\ \frac{2740}{4973} \end{pmatrix}$$

Hence $\sum_{j=0}^{5} jA_j = \sum_{j=0}^{6} B_j$.

Thus, the second condition in (14) is also satisfied.

The consistency conditions are therefore met. Hence, the method is consistent.

Zero stability of the BEBDF method

The stability polynomial of the method (3) is given by:

$$R(t,h) = -\frac{11}{29838} - \frac{6289\,t}{9946} - \frac{3651\,h\,t}{9946} + \frac{211849\,t^2}{9946} + \frac{240933h\,t^2}{9946} + \frac{68922\,h^2\,t^2}{4973} - \frac{616669\,t^3}{29838} + \frac{180249\,h\,t^3}{4973} - \frac{126432\,h^2\,t^3}{4973} + \frac{52560\,h^3\,t^3}{4973} \quad (18)$$

For details, see (Musa *et al*., 2012).

The first characteristics polynomial of the method (3) is given by $(C_0^* t - C_1^*)$ where

$$C_0^* = \begin{pmatrix} 1 & -\frac{25}{16} & -\frac{3}{40} \\ -12 & 1 & \frac{197}{25} \\ \frac{18700}{14919} & -\frac{26550}{14919} & 1 \end{pmatrix}, \quad C_1^* = \begin{pmatrix} -\frac{1}{80} & \frac{1}{8} & -\frac{3}{4} \\ -\frac{3}{25} & 1 & -4 \\ \frac{394}{14919} & -\frac{2925}{14919} & \frac{9600}{14919} \end{pmatrix}$$

Solving $\left|C_0^* - C_1^*\right| = 0,$ the polynomial obtained is:

$$\frac{616669\,t^3}{29838} - \frac{211849\,t^2}{9946} + \frac{6289\,t}{9946} + \frac{11}{29838} = 0 \quad (19)$$

Solving for t gives
t=1, t= –0.000572001, t=0.031184858

Thus, by definition of zero stability, the BEBDF method is zero stable.

Since consistency and zero stability conditions are both satisfied, the fully implicit 3-point BEBDF method converges. This completes the proof of conditions set in the theorem.

NUMERICAL RESULTS

To illustrate the effect of zero stability and consistency on the method, the following non linear problems are solved at some fixed station values of x. The theoretical and numerical results as well as the absolute error for different step length h are given in Tables 1-4.

Problems

1.

$$y' = \frac{y(1-y)}{2y-1}, \qquad y(0) = \frac{5}{6}, \qquad 0 \le x \le 1$$

Exact solution

$$y(x) = \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{5}{36}e^{-x}}$$

Source: (Alvarez *et al*., 2002).

2.

$$y' = -\frac{y^3}{2}, \qquad y(0) = 1, \qquad 0 \le x \le 4$$

Exact solution

$$y(x) = \frac{1}{\sqrt{1+x}}$$

Source: (Voss *et al*., 1997).

Table-1. Effect of zero stability and consistency on the 3-point BEBDF method when problem 1 is solved with h=0.01.

x	Theoretical solution	Numerical solution	Absolute error
0.0	0.8333333	0.8333333	0.0000000
0.1	0.8526020	0.8527450	0.0001430
0.2	0.8691712	0.8690573	0.0001139
0.3	0.8835474	0.8829767	0.0005707
0.4	0.8961060	0.8960859	0.0000201
0.5	0.9071359	0.9065019	0.0006340
0.6	0.9168647	0.9155497	0.0013150
0.7	0.9254760	0.9253058	0.0001702
0.8	0.9331203	0.9321562	0.0009641
0.9	0.9399227	0.9381680	0.0017547
1.0	0.9459884	0.9439650	0.0020234

Table-2. Effect of zero stability and consistency on the 3-point BEBDF method when problem 1 is solved with h=0.001.

x	Theoretical solution	Numerical solution	Absolute error
0.0	0.8333333	0.8333333	0.0000000
0.1	0.8526020	0.8526051	0.0000031
0.2	0.8691712	0.8691327	0.0000385
0.3	0.8835474	0.8834451	0.0001023
0.4	0.8961060	0.8960943	0.0000117
0.5	0.9071359	0.9070486	0.0000873
0.6	0.9168647	0.9166918	0.0001729
0.7	0.9254760	0.9254502	0.0000258
0.8	0.9331203	0.9330021	0.0001182
0.9	0.9399227	0.9397089	0.0002138
1.0	0.9459884	0.9457798	0.0002086

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Table-3. Effect of zero stability and consistency on the 3-point BEBDF method when problem 2 is solved with h=0.01.

x	Theoretical solution	Numerical solution	Absolute error
0	1.0000000	1.0000000	0.0000000
0.1	0.9534626	0.9531365	0.0003261
0.2	0.9128709	0.9131317	0.0002608
0.3	0.8770580	0.8783667	0.0013087
0.4	0.8451543	0.8452111	0.0000568
0.5	0.8164966	0.8180010	0.0015044
⋮	⋮	⋮	⋮
1.0	0.7071068	0.7078439	0.0007371
1.1	0.6900656	0.6930848	0.0030192
1.2	0.6741999	0.6794762	0.0052763
⋮	⋮	⋮	⋮
2.0	0.5773503	0.5814200	0.0040697
2.1	0.5679618	0.5746525	0.0066907
2.2	0.5590170	0.5604601	0.0014431
⋮	⋮	⋮	⋮
3.0	0.5000000	0.5072343	0.0072343
3.1	0.4938648	0.4955359	0.0016711
3.2	0.4879500	0.4925243	0.0045743
⋮	⋮	⋮	⋮
3.8	0.4564355	0.4611000	0.0046645
3.9	0.4517540	0.4591699	0.0074159
4.0	0.4472136	0.4489977	0.0017841

Table-4. Effect of zero stability and consistency on the 3-point BEBDF method when problem 2 is solved with h=0.001.

x	Theoretical solution	Numerical solution	Absolute error
0	1.00000	1.00000	0.0000000
0.1	0.9534626	0.9534550	0.0000076
0.2	0.9128709	0.9129578	0.0000869
0.3	0.8770580	0.8772912	0.0002332
0.4	0.8451543	0.8451819	0.0000276
0.5	0.8164966	0.8167020	0.0002054
⋮	⋮	⋮	⋮
1.0	0.7072001	0.7071068	0.0000933
1.1	0.6900656	0.6904173	0.0003517
1.2	0.6741999	0.6748168	0.0006169
⋮	⋮	⋮	⋮
2.0	0.5773503	0.5778026	0.0004523
2.1	0.5679618	0.5687133	0.0007515
2.2	0.5590170	0.5591786	0.0001616
⋮	⋮	⋮	⋮
3.0	0.5000000	0.5008002	0.0008002
3.1	0.4938648	0.4940479	0.0001831
3.2	0.4879500	0.4884486	0.0004986
⋮	⋮	⋮	⋮
3.8	0.4564355	0.4569413	0.0005058
3.9	0.4517540	0.4525675	0.0008135
4.0	0.4472136	0.4474070	0.0001934

From the above tables, the zero stability of the method is indicated by the decrease in error as the step length h tends to zero. The accuracy also improves as the step length is reduced. Thus, the error is not propagated in any explosive manner.

Similarly, the solution at any fixed point x improves as the step length is reduced. This can be seen when we compare Tables 1 and 2 for problem 1 and Tables 3 and 4 for problem 2.

The absolute error also indicates that the numerical solution becomes close to the exact solution. Thus, the computed solution tends to the theoretical solution as the step length tends to zero. This shows the consistency of the method.

CONCLUSIONS

The paper studied the fully implicit 3-point block extended backward differentiation formula and proved that the method is consistent and zero stable. This indicates that the method is convergent. The numerical results presented illustrated the effect of zero stability and consistency of the method when a stiff IVP is solved. There is no evidence of explosive error propagation in the method. The method was also proven to be of order 6. These added advantages make the BEBDF method to be numerically acceptable method for solving stiff initial value problems.

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