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COMPARISON OF STEHFEST'S AND ISEGER'S ALGORITHMS FOR LAPLACIAN INVERSION IN PRESSURE WELL TESTS

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ABSTRACT

This paper presents the comparison of two algorithms used for the numerical inversion of the Laplace transform. The comparison was applied to two well-known oil-industry reservoir models in the Laplacian domain, for which the inversion is made and the results are plotted to establish comparison. Not only accuracy but also computing effort was studied. Although, the Iseger's algorithm is computationally much heavier it handles more efficiently functions with discontinuities or functions having sharp changes. It was also observed in well tests that fulfilling the conditions relating time with the number of sample points leads to more stable inversions. It was also found that the greater the number of points to be inverted, the more accurate the solution. Moreover, the oversampling parameter *nrp* provides more stable solutions when it takes the value of three contrary to eight as initially proposed by Iseger.

Keywords: laplace, numerical inversion, homogeneous reservoirs, heterogeneous reservoirs.

1. INTRODUCTION

Since the work presented by van Everdingen and Hurst (1953), the Laplace transform has been a standard tool for solving transient problems for fluid flow in porous materials. The Laplace transform is mainly used for the solution of initial boundary value problems. For some of these applications, an exact analytical inversion is not possible and then numerical inversion is the only resource. For some others, the numerical inversion is also chosen for its convenience. For this inversion, the Stehfest's algorithm (1970) is normally used and included in conventional software applications in the oil industry. However, it has such limitations as handling of discontinuous functions and stiff functions which are better handled by the Iseger's algorithm, Al-Ajmi *et al*. (2008).

For numerical inversion of Laplace Transform several algorithms have been proposed. Among them, we can name the works presented by Bellman, Kalaba and Lockett (1966), Crump (1976), Kryzhniy (2004) and Talbot (1979). However, they did not have much acceptance. The most famous method in the oil industry is the Stehfest's algorithm (1970). However, it is not very accurate when handling either sharply changing functions such the situation presented in the trough found on the pressure derivative behavior of naturally-fractured formations when the transition period appears just after wellbore storage vanishes or the inverse Laplace in singular/discontinuous functions. For these cases, Iseger's algorithm (2006) alleviates this type of restrictions and provides opportunities for many practical applications.

2. METHODOLOGY

2.1. Election of the programming tool and reservoir models

Visual Basic 2008 was used as the programming environment for the easiness on handling plots as well as the accuracy. However, Iseger's algorith works in the plane of complex numbers. Since Visual Basic 2008 does not possess management tools for working with complex numbers, it was necessary to use some external libraries that include all needed functions for the declaration and operation of high precision complex variables.

Models for homogeneous and heterogeneous infinite reservoirs were chosen to test the kindness, advantages and drawbacks of Iseger's and Stehfest's algorithms. The model for heterogeneous reservoirs was the one proposed by Warren and Root (1963).

2.2. Iseger's algorithm

This algorithm has the property of eliminating the continuity restriction presented in some of the common inversion algorithms. This functionality makes the Iseger's algorithm to be applied for a wider range of applications.

The algorithm proposed by Iseger (2006) is a Fourier series method based upon the Poisson summation formula. The summation of Poisson relates an infinite sum of Laplace transform values with the *Z* transformation values of the function. The infinite sum is approximated by a finite sum based on the Gaussian Quadrature rule and the values in the time domain of the function are calculated by a Fourier transform algorithm. The results offered are related to the precision of the machine, Al-Ajmi *et al*. (2008).

The attractiveness of the Iseger's algorithm lies on its ability to calculate the inverse of the Laplace transform for functions with all kinds of discontinuities, singularities and without local smoothing, although it should be pointed out that the implementation of this algorithm is more complicated than the implementation of the Stehfest's algorithm, but it is comparable to the implementation of other common algorithms.

There are several parameters to be taken into account in the Iseger's algorithm since they affect the accuracy of the results. A critical parameter used is Δ

(delta) which can be calculated by the following relationship:

$$\Delta = \frac{T}{M} \tag{1}$$

Where T is the period for which the inversions are calculated and M is the number of points for which is calculated the Laplace inverse.

The approximation error also depends on Δ, to guarantee accurate and reliable results, two conditions must be met. The first condition to be fulfilled is that $(1/\Delta)$

$\in N$. As noted by Al-Ajmi *et al*. (2008), there must also

meet that,

$M \geq 10\, \Delta M$, which is equivalent to $M \geq 10\, T$

By fulfilling these conditions, stability and accuracy results are guaranteed in the inversion of the Laplace transform. It is important to note that the greater the number of sample points the higher the accuracy; then, needless to say that in order to obtain stable and accurate inversions M has to to be as large as possible.

Iseger's algorithm (2006) uses M_2=*nrp* (M) oversampling points to calculate the inversions en M data points. The actual execution time is $M_2\log(M_2)$; therefore, from this perspective, M_2 ought to be chosen as small as possible. However, according to Al-Ajmi et al. (2008) in obtaining numerically stable results, M_2 should be chosen as large as possible.

The *nrp* parameter, given by Iseger (2006), does not have a formal definition but it is called the oversampling factor, which increases the accuracy of the results. Iseger (2006) recommends using $M_2 = 8M$, which translates into *nrp* = 8, particularly for well-behaved functions. This particular choice for *nrp* was because of the conditions imposed by the Fast Fourier Transform as originally used by Iseger, which requires *nrp* and M to be a power of 2 (iseger, 2006).

In the Iseger's algorithm, the parameter n was arbitrarily defined as n = 16; therefore, constants beta and lambda for this value of n were defined according to the values provided by him - Iseger (2006).

n = 16 for all smooth functions
n = 32 for very oscillating singular functions
n = 48 when the function has many peaks

Iseger (2006) shows several algorithms with various modifications. The model used here is the most complete, which means, it has wider applications. It was proved to handle discontinuities of all kind without any problem even without considering the location of discontinuity; therefore, as far as handling discontinuities, its confidence is unlimited.

2.3. Comparison of algorithms' performance

Several cases are shown here in which the stability and accuracy of the Iseger algorithm could be affected.

A unit-step function with a testing period T of 20 hr was considered. Based on the before mentioned conditions, the number of sampling points must $M \geq 200$. Then, to observe the effect of the number of inversion points on the results, three different M values were used.

According to Al-Ajmi *et al*. (2008), Figure-1 shows unappropriated Laplace inversions using 103 and 233 data points. The function losses its nature at the discontinuity point since 103 data points do not meet any of the required conditions while 233 points just meet one. The desire inversion is gotten when using 220 points since both conditions are fulfilled.

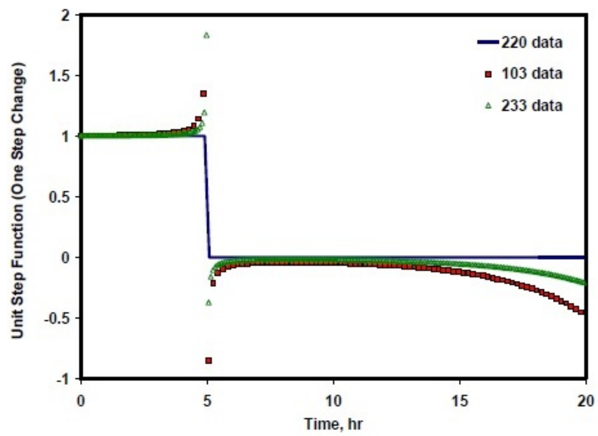

Figure-1. Numerical inversión of a unit-setp function by the Iseger's Algoritm. After Al-Ajmi *et al*. (2008).

Further from Al-Ajmi *et al*. (2008), applications of Iseger's algorithm for the numerical inversion of fluid flow models do not provide stable solutions for *nrp = 8*. Stable inversions were obtained by trial-and-error using discrete Fourier transforms and *nrp* = 3. Figure-2 shows the effect of *nrp* on the numerical inversion found in mentioned models of well test pressure behavior by means of the algorithm proposed by Iseger.

Several functions were used for testing the validity of our computer program with an arbitrary period T = 20 and M = 220 inversion points to fulfill the inversion stability. As examples, Figure-3 reports the inversion solution for a multi-step function and Figure-4 for inversion of the Sine function.

Both Stehfest's and Iseger's algorithm were tested for inverting the pressure behavior of a well in an infinite reservoir. Since the behaviors are very close, only the absolute dimensionless pressure error is reported in Figure-5 having the Iseger's algorithm as the reference point.

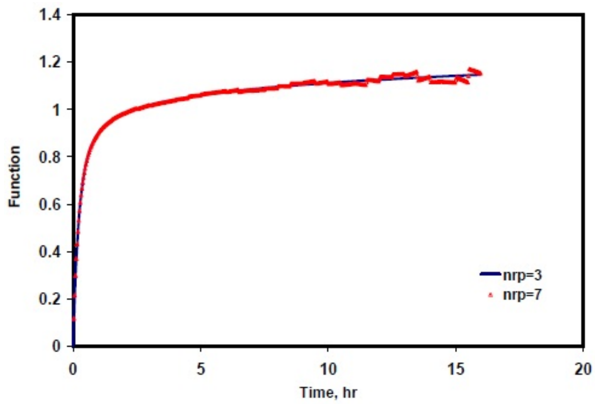

Figure-2. Effect of *nrp* in the numerical inversion using the Iseger's algorithm. After Al-Ajmi *et al*. (2008).

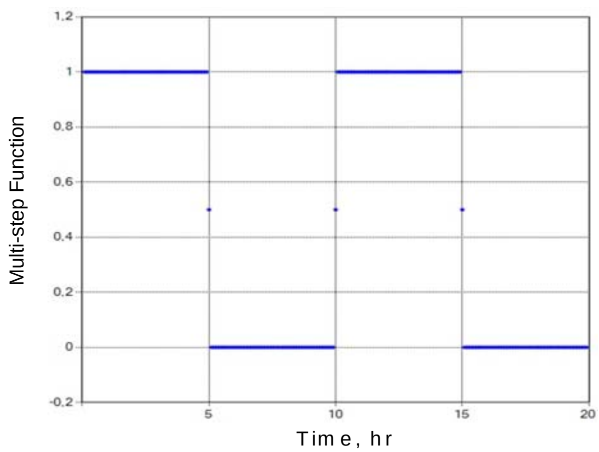

Figure-3. Inversion of a multi-step function using the Iseger's algorithm.

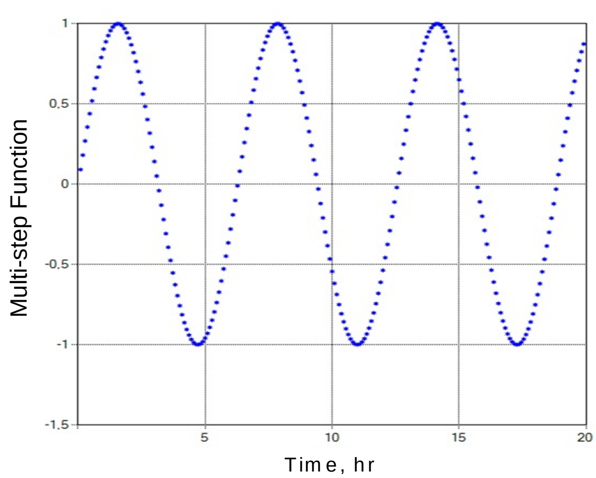

Figure-4. Inversion of the Sine function using the Iseger's algorithm.

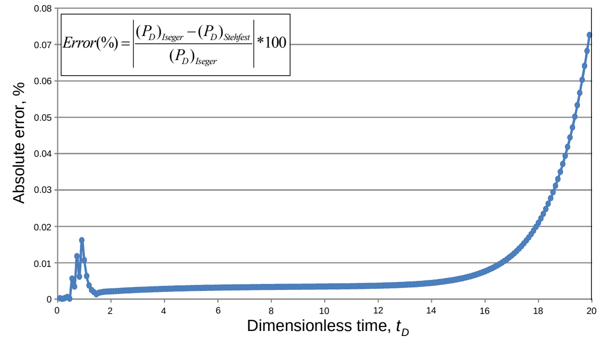

Figure-5. Absolute error of the dimensionless pressure found by inverting with Iseger's and Stehfest's algorithm for a well in an infinite homogeneous reservoir.

Figure-5 allows noting a small difference between the two worked inversion methods. There is a small data distortion at the beginning and then a slight stabilization and an increasing tendency of the difference which can be due to round-off error accumulation. However, since the difference is very small we can conclude that both methods are equally applied.

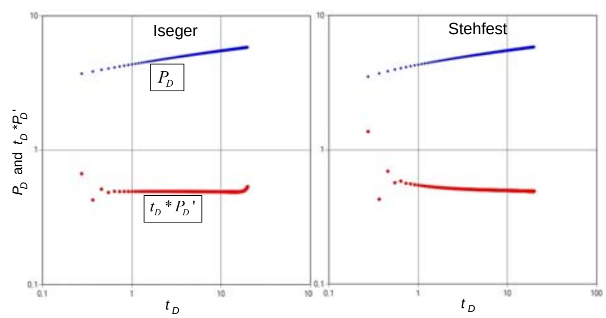

Figure-6. Dimensionless pressure and pressure derivative vs. time behavior for a well in an infinite heterogeneous reservoir using Iseger's and Stehfest's algorithm for Laplace inversion.

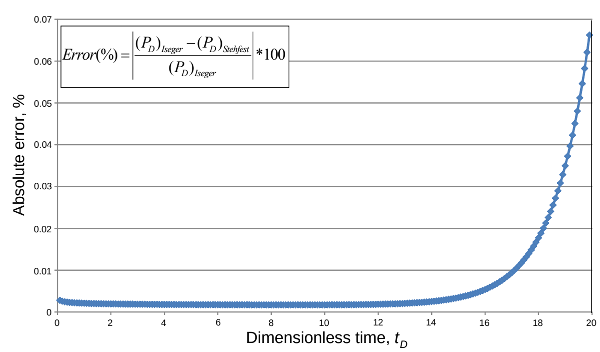

Figure-7. Absolute error of the dimensionless pressure found by inverting with Iseger's and Stehfest's algorithm for a well in an infinite heterogeneous reservoir.

Figure-6 shows the dimensionless pressure and pressure derivative response for a naturally-fractured reservoir using the Warren-and-Root (1963) parameters: $\lambda = 0.000001$ and $\omega = 0.001$. As seen in Figure-7, the dimensionless pressure differences are small, although, as expected, a higher early distortion is given by the Stehfest's algorithm. Also, the pressure derivative shows a less stable behavior using Stehfest, maybe, caused by error accumulation.

3. CONCLUSIONS

It was observed that discontinuities are better handled while inverting Laplace domain solutions by using the algorithm developed by Iseger than the commonly used Stehfest algorithm. Although, for oil-field applications the Iseger algorithm provides better results in stiff functions, however its manipulation is much more complicated than the Stehfest's algorithm and also machine time computation increases. This, however, is not a big problem with today's computers.

Nomenclature

n	Parameter depending on the function type
nrp	Oversampling factor
M	Number of inverting points
P_D	Dimensionless pressure
t_D	Dimensionless time
T	Period

Greek

Δ	Ratio T/M
λ	Interporosity flow parameter
ω	Dimensionless storativity ratio

While in the Stehfest's algorithm, the number of sample points is the same number of calculated inversions since time is handled in a logarithmic way; in Iseger's algorithm is necessary to take into account the needed conditions to be met to obtain reliable and stable results, so the period and the number of points are directly proportional and depends of the parameter Δ. The condition indicates that the number of sampled points must be at least 10 times greater than the period for which inversions are made. Because of this, the number of calculations for Iseger's algorithm is about 10 times greater than in the algorithm Stehfest, so it is not as suitable for large periods.

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