ARPN Journal of Engineering and Applied Sciences

www.arpnjournals.com

CATEGORIZATION OF NORMAL SUB LOOP AND IDEAL OF LOOPS

V. B. V. N. Prasad[1] and J. Venkateswara Rao[2]
[1]Department of Mathematics, Vignan Degree College, (Affiliated to Acharya Nagarjuna University), Guntur, AP, India
[2]Department of Mathematics, Mekelle University, Mekelle, Ethiopia
E-Mail: bhanuvarikuti@yahoo.co.in

ABSTRACT

This manuscript illustrates the significance of loops, sub loops, normal sub loop and ideal of loops, when compared to general groups and subgroups. It also distinguishes the relations between normal sub loops and ideal of loops. Further various properties are verified on loops, normal sub loop and ideal of loops when they are compared to groups and subgroups. Various characteristics of normal sub loops and ideal of loops were obtained in additive notation also.

Keywords: ideals, congruence relations, loops, quasigroups, convexity, generalizations.

1. INTRODUCTION

Garrett Birkhoff (1942) firstly initiated the notion of lattice ordered groups. Then Bruck (1944) contributed various results in the theory of quasigroups. Zelinski (1948) described about ordered loops. The concept of non associative number theory was thoroughly studied by Evans (1957). Bruck (1963) explained about what is a loop? Various crucial properties of lattice ordered groups were established by Garrett Birkhoff in 1964 and 1967. Evans (1970) described about lattice ordered loops and quasigroups. Richard Hubert Bruck (1971) made a survey of binary systems. In the recent past Hala (1990) made a description on quasigroups and loops.

In this document we furnish definitions, examples and some properties of Normal sub loops and ideal of loops when compared with groups, in additive notation. In this manuscript mainly there are two topics, one is about normal sub loops and the other is about ideal of a loop, and the definitions, examples and properties are in additive notation. Here we provide some of the following foremost properties:

a) A sub loop of a loop is normal if and only if it is invariant under all inner mappings of the loop.
b) Every inner mapping is an order automorphism
c) Let θ be a congruence relation on $\mathcal{L}$. Then $\mathcal{N}_\theta$ = {a$\in \mathcal{L} / a \equiv 0(mod\theta)\}$ is an ideal of $\mathcal{L}$.
d) Let $\mathcal{N}$ be an ideal $of\ \mathcal{L}.$ Define $\theta_{\mathcal{N}}$ on $\mathcal{L}$ as a $\equiv$ (mod$\theta_{\mathcal{N}}$) iff a-b$\in \mathcal{N}$. Then θ is a congruence relation on $\mathcal{L}$.
e) For any ideal $\mathcal{N}$ of $\mathcal{L}$, $\mathcal{N}$ = $\mathcal{N}_{\theta_{\mathcal{N}}}$, for any congruence relation θ on $\mathcal{L}$, θ = θ ($\mathcal{N}_\theta$).

2. NORMAL SUBLOOP AND IDEALOF LOOPS

Definition (2.1): A loop is a quasigroup (S, +) with two sided identity '0' satisfying 0+x=x+0=x for all x in S.

Note (2.1): It follows that the identity element '0' is unique and that every element of S has unique left and right inverse.

Example (2.1): Every group is a loop, because a+x=b if and only if x = (-a) +b and y+a=b if and only if y=b + (-a)

Note (2.2): In a loop x/x=x-x=0 and x\x= - x+x=0 for any x

Definition (2.2) A system (S, +, \, /), where S is a nonempty set and +, \, / are binary operations on S satisfying the following identities:

(i) a+ (a\c) = c and (c/b) + b = c,

(ii) a\ (a+b) = b and (a+b)/b = a,

(iii) c/ (a\c) = a and (c/b)\c = b, for all a, b, c in S, is called an equasigroup.

Note (2.3): Let (S, +) be a quasigroup. If we define a\b = - a + b, a/b = a – b then (S, +, \, /) is an equasigroup.

Proof: Let (S, +) be a quasigroup.

(i) a + (a\c) = a + (- a + c) = aa + c = c

and (c/b) + b = c – b + b = c.

(ii) a\(a + b) = -a+ a + b = b

and (a + b)/b = a + b – b = a

(iii) c/ (a\c) = c – (a\c) = c – (- a + c) = c + a – c = a

VOL. 9, NO. 7, JULY 2014 ISSN 1819-6608
ARPN Journal of Engineering and Applied Sciences

and (c/b)\c = (c – b)\c = - (c –b) + c

= - c + b + c = b.

Therefore (S, +, \, /) is an equasigroup.

Note (2.4): Every equasigroup is a quasigroup.

Proof: Let (S, +, \, /) is an equasigroup.

Therefore S satisfies the following identities.

(i) a + (a\c) = c and (c/b) + b = c.

(ii) a\ (a+b) = b and (a+b) /b = a.

(iii) c/(a\c) = a and (c/b)\c = b

The first identity state that the equations a + x = c, y + b =c have solutions and the second shows the uniqueness. Hence (S, +) is a quasigroup.

Note (2.5): From Note (2.3) and Note (2.4), every quasigroup is equationally definable.

Note (2.6): A loop is equationally definable.

Proof: By Note (2.5) and the fact that x+0=0+x=x for any x. Hence we have that any loop is equationally definable.

Definition (2.2): Let L is a loop. A non-empty subset H of L is called a Sub loop of L if H itself is a loop under the operation of L.

Definition (2.3): A sub loop $\mathcal{N}$ of a loop $\mathcal{L}$ is normal if for all x, y in $\mathcal{L}$

(1) x +$\mathcal{N}$ = $\mathcal{N}$ + x
(2) x + (y + $\mathcal{N}$) = (x + y) + $\mathcal{N}$
(3) x + ($\mathcal{N}$ + y) = (x + $\mathcal{N}$) +y
(4) ($\mathcal{N}$ + x) + y = $\mathcal{N}$ + (x + y)

Definition (2.4): A loop L is said to be a simple loop if it does not contain any non-trivial normal sub loop.

Definition (2.5): The left addition mapping λ_a by an element a of $\mathcal{L}$ is defined by λ_a: x $\rightarrow$ a + x for all x in $\mathcal{L}$.

Definition (2.6): The right addition mapping ρ_a by an element a of $\mathcal{L}$ is defined by ρ_a: x $\mapsto$ x + a for all x in $\mathcal{L}$.

Definition (2.7): By an inner mapping of the loop $\mathcal{L}$ we mean any mapping of the form $\lambda_a \rho_a^{-1}$; $\lambda_a \lambda_b \lambda^{-1}_{b+a}$; $\rho_a \rho_b \rho_{a+b}^{-1}$; a, b in $\mathcal{L}$.

Note (2.7): A sub loop of a loop is normal if and only if it is invariant under all inner mappings of the loop.

Note (2.8): Every inner mapping is an order automorphism. That is x$\leq$ y iff θ (x) $\leq$ θ (y) and θ (x $\vee$ y) = θ (x) $\vee$ θ (y), θ (x $\wedge$ y) =θ (x) $\wedge$$\theta$ (y), for all elements x, y and all inner mappings θ.

Definition (2.8): Let L is a lattice ordered loop. A sub set $\mathcal{C}$ of $\mathcal{L}$ is said to be convex if x $\in \mathcal{C}$ whenever a, b$\in$ $\mathcal{C}$ and a$\leq$ x$\leq$ b.

Definition (2.9): A normal sub loop of a loop ($\mathcal{L}$, +) which is also a convex sub lattice of ($\mathcal{L}$, $\wedge$, $\vee$) is called an ideal of $\mathcal{L}$.

Theorem (2.1): Let $\mathcal{L}$ be a lattice ordered loop.
(i) Let θ be a congruence relation on $\mathcal{L}$. Then $\mathcal{N}_\theta$ = {a$\in$ $\mathcal{L}$ /a $\equiv$ 0 (mod θ)} is an ideal of $\mathcal{L}$.
(ii) Let $\mathcal{N}$ be an ideal of $\mathcal{L}$. Define $\theta_\mathcal{N}$ on $\mathcal{L}$ as a $\equiv$ b (mod$\theta_\mathcal{N}$) iff a-b$\in$ $\mathcal{N}$ .Then θ is a congruence relation on $\mathcal{L}$
(iii) For any ideal $\mathcal{N}$ of $\mathcal{L}$, $\mathcal{N}$ = $\mathcal{N}_{\theta_\mathcal{N}}$; for any congruence relation θ on $\mathcal{L}$, θ = θ ($\mathcal{N}_\theta$).

Proof: (i) Let θ be a congruence relation on $\mathcal{L}$.
Now we prove that

$$\mathcal{N}_\theta = \left\{a \in \frac{\mathcal{L}}{a} \equiv 0 \ (\mathrm{mod}\ \theta)\right\} \textbf{is an ideal of } \mathcal{L}.$$

First we prove that $\mathcal{N}_\theta$ is a sub loop of $\mathcal{L}$
Let a, b$\in$ $\mathcal{N}_\theta$
So a $\equiv$ 0 (modθ), b$\equiv$ 0 (modθ)
Since θ is a congruence relation on $\mathcal{L}$
a+b $\equiv$ 0+0 =0 (modθ)
a\b = 0\0 =0 (modθ) and a/b = 0/0 = 0(modθ)
So a+b, a\b, a/b are in $\mathcal{N}_\theta$
Clearly 0$\in$$\mathcal{N}_\theta$
So $\mathcal{N}_\theta$ is a sub loop of $\mathcal{L}$.

Now we prove that $\mathcal{N}_\theta$ is normal:
Let x$\in$ $\mathcal{L}$
We prove that (1) x+$\mathcal{N}_\theta$ = $\mathcal{N}_\theta$ + x
Let y $\in$ $\mathcal{N}_\theta$

ARPN Journal of Engineering and Applied Sciences

www.arpnjournals.com

Put Z = (x+y) / x
Y $\equiv$ 0(θ) $\Rightarrow$ x+y $\equiv$ (x+o) (θ)
$\Rightarrow$ x+y$\equiv$ x (θ)
$\Rightarrow$ (x+y)/ x $\equiv$ x/x (θ)
That is Z $\equiv$ 0 (θ) i.e. Z $\in$ $\mathcal{N}_\theta$
x + y = (x + y) /x + x (*since* a/c + c = a)
= z + x $\in$ $\mathcal{N}_\theta$ + x.
Therefore x + $\mathcal{N}_\theta$ $\subseteq$ $\mathcal{N}_\theta$ +x.
Similarly $\mathcal{N}_\theta$ +x $\subseteq$ x + $\mathcal{N}_\theta$
Therefore x + $\mathcal{N}_\theta$ = $\mathcal{N}_\theta$ +x.

(2) Let x, y $\in$ $\mathcal{L}$. Let z $\in$ $\mathcal{N}_\theta$
x + (y+z) = (x+y) + p for some p $\in$ $\mathcal{N}_\theta$
Write p =(x+y)\[x+(y+z)]
z$\equiv$ 0(θ)$\Rightarrow$ y+z $\equiv$ y (θ)
$\Rightarrow$ x+(y+z) = x+y(θ)
$\Rightarrow$ (x+y)\[x+(y+z)] =(x+y)\(x+y) (θ) =0(θ), where p=(x+y)\[x+(y+z)]
$\Rightarrow$ p $\in$ $\mathcal{N}_\theta$.
Clearly x+(y+z) =(x+y) +p$\in$ (x+y) + $\mathcal{N}_\theta$.
Therefore x+(y+$\mathcal{N}_\theta$) $\subseteq$ (x+y) + $\mathcal{N}_\theta$ $\rightarrow$ (I)
Let z$\in$ $\mathcal{N}_\theta$
Put p=y\ ((x+y) +z))
So y+p=x\ ((x+y) +z)
So (x+y) +z=x+(y+p) $\in$ x+(y+$\mathcal{N}_\theta$)
z$\in$ $\mathcal{N}_\theta$ $\Rightarrow$ z$\equiv$ 0(θ)
$\Rightarrow$ (x+y)+z=(x+y)(θ)
$\Rightarrow$ x\ [(x+y) +z]$\equiv$ x\(x+y)(θ)$\equiv$ y(θ)
$\Rightarrow$ y\ (x\ ((x+y) +z)) =y\y (θ) =0(θ)
That is p$\equiv$ 0(θ)
Therefore p$\in$ $\mathcal{N}_\theta$.
Therefore (x+y) + $\mathcal{N}_\theta$ $\subseteq$ x+(y+$\mathcal{N}_\theta$) $\rightarrow$ (II)
Therefore from (I) and (II) (x+y) + $\mathcal{N}_\theta$ = x+(y+$\mathcal{N}_\theta$).

(3)To prove that x+ ($\mathcal{N}_\theta$ +y) =(x+$\mathcal{N}_\theta$) +y.
Let z$\in$ $\mathcal{N}_\theta$
Put p=x\ {[x+ (z+y)]/y}
Therefore x+p = [x+ (z+y)]/y
Therefore (x+p) +y=x+ (z+y)
Now p=x\ {[x+ (0+y)]/y} (modθ) ($\because$ z$\in$ $\mathcal{N}_\theta$)
=x\ {(x+y)/y} (modθ)
=x\x (modθ)
=0(modθ)
Therefore p $\in$ $\mathcal{N}_\theta$.
Therefore x+ (z+y) =(x+p) +y $\in$ (z+$\mathcal{N}_\theta$) +y
Therefore x+ ($\mathcal{N}_\theta$ +y) $\subseteq$ (x+$\mathcal{N}_\theta$) +y
Similarly (x+$\mathcal{N}_\theta$) + y $\subseteq$ x + ($\mathcal{N}_\theta$ +y)
Therefore x+ ($\mathcal{N}_\theta$ +y) = (x+$\mathcal{N}_\theta$) +y.

(4)($\mathcal{N}_\theta$ +x)+y=$\mathcal{N}_\theta$ +(x+y)
Let z$\in$ $\mathcal{N}_\theta$
(z+x)+y=z+(x+y)
Put [(z+x) +y]/(x+y) =p
Clearly (z+x) +y=p+(x+y)
p$\equiv$ [(0+x)+y]/x+y=(x+y)/(x+y)=0(modp)
Therefore p$\in$ $\mathcal{N}_\theta$
Therefore ($\mathcal{N}_\theta$ +x) +y$\subseteq$ $\mathcal{N}_\theta$ +(x+y)
Similarly [($\mathcal{N}$□$_1\theta$ +x) +y $\supseteq$ $\mathcal{N}_\theta$ +(x+y)
□**Therefore** ($\mathcal{N}$□$_1\theta$ +x) +y = $\mathcal{N}_\theta$ +(x+y)
Now we can prove that $\mathcal{N}_\theta$ is convex:
Let a, b $\in$ $\mathcal{N}_\theta$, x$\in$ $\mathcal{L}$ such that a$\wedge$ b$\leq$ x$\leq$ a$\vee$ b
Now a, b$\in$ $\mathcal{N}_\theta$ $\Rightarrow$ a$\equiv$ 0(θ), b$\equiv$ 0(θ)
a$\wedge$ b$\equiv$ 0$\wedge$ 0(θ)$\equiv$ 0(θ)
a$\vee$ b$\equiv$ 0(θ)
a$\wedge$ b$\equiv$ a$\vee$ b (θ)
(a$\wedge$ b)$\vee$ x $\equiv$ (a$\vee$ b)$\vee$ x (θ)
$\Rightarrow$ x$\equiv$ (a$\vee$ b) (θ)$\equiv$ 0(θ)
$\Rightarrow$ $x \in \mathcal{N}_\theta$
So $\mathcal{N}_\theta$ is a convex sub lattice.
(ii) Let $\mathcal{N}$ *be an ideal of* $\mathcal{L}$
Define $\theta_\mathcal{N}$ on $\mathcal{L}$ as a$\equiv$ b (mod$\theta_\mathcal{N}$) iff a-b$\in$ $\mathcal{N}$
Now we prove that $\theta_\mathcal{N}$ is a congruence relation on $\mathcal{L}$
For this first we prove that $\theta_\mathcal{N}$ is an equivalence relation on $\mathcal{L}$

i) Reflexive: For a$\in$ $\mathcal{L}$, a-a=0$\in$ $\mathcal{N}$ then a$\equiv$ a(mod$\theta_\mathcal{N}$)

(ii) Symmetric: Let a, b$\in$ $\mathcal{L}$
Suppose a$\equiv$ b (mod$\theta_\mathcal{N}$)
That is a-b$\in$ $\mathcal{N}$
That is - (a-b) $\in$ $\mathcal{N}$
That is b-a$\in$ $\mathcal{N}$ i.e. b$\equiv$ a (mod$\theta_1\mathcal{N}$)

(iii) Transitive: Let a, b$\in$ $\mathcal{L}$
Suppose a$\equiv$ b (mod$\theta_\mathcal{N}$) and b$\equiv$ c(mod$\theta_\mathcal{N}$)
$\Rightarrow$ a-b$\in$ $\mathcal{N}$ and b-c$\in$ $\mathcal{N}$
$\Rightarrow$ (a-b) + (b-c) $\in$ $\mathcal{N}$
$\Rightarrow$ a-c $\in$ $\mathcal{N}$
That is a$\equiv$ c (mod$\theta_\mathcal{N}$).

Therefore $\theta_\mathcal{N}$ is an equivalence relation on $\mathcal{L}$.

To prove that $\theta_\mathcal{N}$ is a congruence relation on $\mathcal{L}$
Claim: a+$a_1$$\equiv$ (b+b_1) (mod$\theta_\mathcal{N}$)
a\$a_1$$\equiv$ (b\$b_1$) (mod$\theta_\mathcal{N}$)
a/$a_1$$\equiv$ (b/b_1) (mod$\theta_\mathcal{N}$)
(a+a_1)- (b+b_1) $\in$ $\mathcal{N}$
(a+a_1)- (b+b_1)= $p \in \mathcal{N}$
(a+a_1)= (b+b_1) +p for some $p \in \mathcal{N}$
Now a$\in$ $\mathcal{N}$ +b, $a_1$$\in$ $\mathcal{N}$ +b_1
$\Rightarrow$ a+$a_1$$\in$ ($\mathcal{N}$ +b) + ($\mathcal{N}$ +b_1) $\in$ $\mathcal{N}$ + (b+b_1)
That is (a+a_1)-(b+b_1) $\in$ $\mathcal{N}$
That is (a+a_1) $\equiv$ (b+b_1) (mod$\theta_\mathcal{N}$).

(iii) Let $\mathcal{N}$ be an ideal of $\mathcal{L}$
Let x $\in \mathcal{N}$ $\Rightarrow$ x-0=x$\in \mathcal{N}$
$\Rightarrow$ x $\equiv$ 0(mod$\theta_{\mathcal{N}}$)
$\Rightarrow$ x $\in$ $\mathcal{N}_{I(\theta]_{\mathcal{N}})}$
Therefore $\mathcal{N} \subseteq \mathcal{N}_{I(\theta]_{\mathcal{N}})}$
Let x $\in \mathcal{N}_{I(\theta]_{\mathcal{N}})}$
$\Rightarrow$ x $\equiv$ 0(mod$\theta_{\mathcal{N}}$)
$\Rightarrow$ x-0=x$\in$ $\mathcal{N}$
Therefore $\mathcal{N}_{I(\theta]_{\mathcal{N}})} \subseteq \mathcal{N}$
Therefore $\mathcal{N}$ = $\mathcal{N}_{I(\theta]_{\mathcal{N}})}$.

Let θ be a congruence relation on $\mathcal{L}$.
a $\equiv$ b (modθ)
$\Rightarrow$ a-b $\in \mathcal{N}_\theta$
$\Rightarrow$ a$\equiv$ b (mod$\theta_{I(\mathcal{N}]_\theta)}$)
a$\equiv$ b (mod$\theta_{I(\mathcal{N}]_\theta)}$) $\Rightarrow$ a-b $\in \mathcal{N}_\theta$
$\Rightarrow$ a-b$\equiv$ 0(modθ)
$\Rightarrow$ (a-b)+b=0+b (modθ)
$\Rightarrow$ a$\equiv$ $b(mod\ \ \theta)$
Therefore $\theta =$ $\theta_{I(\mathcal{N}]_\theta)}$.
Hence the proof.

3. Conclusion:

This research work make possible that a sub loop of a loop is normal if and only if it is invariant under all inner mappings of the loop. Every inner mapping is an order automorphism. If θ is a congruence relation on $\mathcal{L}$, then it is observed that $\mathcal{N}_\theta$ = {a$\in$ $\mathcal{L}$ /a $\equiv$ 0 (mod θ)} is an ideal of $\mathcal{L}$. If $\mathcal{N}$ is an ideal $of\ \mathcal{L}$, then there is defined $\theta_{\mathcal{N}}$ on $\mathcal{L}$ as a $\equiv$ b (mod$\theta_{\mathcal{N}}$) iff a-b$\in$ $\mathcal{N}$. Then θ is a congruence relation on $\mathcal{L}$. For any ideal $\mathcal{N}$ of $\mathcal{L}$, $\mathcal{N}$ = $\mathcal{N}_{\theta_{\mathcal{N}}}$; for any congruence relation θ on $\mathcal{L}$, θ = θ ($\mathcal{N}_\theta$). In this manuscript mainly there are two topics, one is about normal sub loops and the other is about ideal of a loop, and the definitions, examples and properties are in additive notation.

REFERENCES

D. Zelinski. 1948. On Ordered Loops, Amer. J. Math. 70: 681-697.

G. Birkhoff. 1942. Lattice Ordered Groups. Ann. of Math. 43: 298-331.

G. Birkhoff. 1967. Lattice Theory, 3rd ed. Amer. Math. Soc. Colloq. Publi.

Hala O. Pflugfelder. 1990. Quasigroups and loops, Heldermann Verlag, Sigma series in Pure Mathematics. 7: 28-59

J. Aczel. 1965. Quasi-groups, nets and nomograms. Adv. Math. 1(3): 383-450.

L. Fuchs. 1963. Partially ordered Algebraic Systems. Addison-Wesley.

R.H. Bruck. 1958. A Survey of Binary Systems, Ergeh, der Math.

Richard Hubert Bruck. 1971. A Survey of binary systems, Springer, 3rd edition. Reprint, Ergebnisse der Mathematik and ihrer Grenz gebiete. 20: 23-167.

T. Evans. 1957. Non-Associative number Theory, Amer. Math. Monthly. 64(5): 299-309.

T. Evans. 1970. Lattice ordered loops and quasigroups. Journal of Algebra. 16: 218-22.

V.B.V.N.Prasad, J.Venkateswara Rao, “Characterization of Quasigroups and Loops”, International Journal of Scientific and Innovative Mathematical Research” (IJSIMR), Volume 1, Issue 2, October – 2013, PP 95-102.

V.B.V.N.Prasad, J.Venkateswara Rao, “Classification of Partial ordered Loops and Lattice Ordered Loops”, International Journal of Mathematical Archive-4(10), 2013, PP.78-84.

V.BV.N.Prasad, J.Venkateswara Rao, “Characteristics of Atoms in Lattice Ordered Loops”, International Journal of Modern Science and Engineering Technology (IJMSET), Volume1, Issue 1, 2014, pp.16-22.

V.B.V.N.Prasad, J.Venkateswara Rao, “Cones in Lattice Ordered Loops”, International Journal of Mathematics and Computer Applications Research (IJMCAR), (Accepted for publication), 2014, (ISSN (print): 2249-6955, ISSN (online): 2249-8060).

W.B. Vasantha Kandasamy. 2002. Smarandache loops. pp. 15-40.