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METHODOLOGY FOR SOLVING THE DIVERGENCE OF FIXED POINT METHOD FOR THE SOLUTION OF A NONLINEAR EQUATION

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ABSTRACT

When we need to determine the solution of a nonlinear equation, there are two options for doing: (a) "closed-methods" which use intervals that contain the root and during the iterative process reduce the size of "smart" way, and, (b) "open-methods" which represent an attractive option as they don't require an initial interval enclosure. In general, we know open-methods are more efficient computationally though don't always work suitably. In this paper we are presenting the study of a very particular divergence case when we use open-methods, in fact, we use the method of fixed point iteration to look for square roots. To solve this problem, we propose to apply some tricks (developed by authors) to modify the iteration function. We propose two alternatives doing additional formulations of the traditional method and its convergence theorem. Although the situation has been studied with other methods like Newton an interesting divergence situation is presented in the method of fixed point iteration which probably could be solved by using another method, however the goal here is to demonstrate that this situation can be solved and additionally is possible to get a convergence higher than quadratic convergence in the first iterations when we use the proposed alternatives.

Keywords: divergence, fixed point, linear convergence, open methods, quadratic convergence, root equations.

1. INTRODUCTION

When we want to determine the roots or zeros of an equation, that is, values of *x* which cause f(*x*)=0, is possible using two ways: analytical direct methods that are restricted to particular cases such as classical quadratic equation, or, numerical methods which covering a broad spectrum, for example, solving algebraic equations, transcendental and polynomial. In general, there are two philosophies work for finding roots of equations: closed and open methods (Akai, 2004) (Nakamura, 1997) (Chapra and Canale, 2007).

Intervals that enclose or contain the root are used by closed methods. These methods reduce the work interval using a particular criterion for each method; such is the case of the bisection methods and false position or false rule. These methods perform so well, however, we know their convergence is too slow even in some cases is deficient. Therefore, we cannot generalize about the benefits of a method over another. Rather than have a large battery of methods that can be used in case of failure or improper behavior of convergence. There is also the possibility of finding new methods including modifications to existing.

Iterations of closed methods always generate approximations ever closer to the root; therefore, we say they are convergent because they are progressively closer to the root as they advance calculation cycles. Meanwhile, the open methods are based on iteration formulas requiring only a starting point or pair of values that need not necessarily enclose the root. This quality provides important advantages; however, a difficulty arises that is related to the divergence of the methods, so it is necessary to plan alternatives to face the problem, especially when we know it's worth doing, since, in general, when the open methods converge, they do so more quickly than closed methods (Press, Teukolski, and Vetterling, 2012).

Open methods using a general strategy of successive substitutions. Examples of these methods are Newton, Secant and the method of fixed point iteration. On this last method the attention of this article focuses, as there is a whole mystique around the divergence of this method and the alternatives for improvement (Heath, 2002). Perhaps the easiest way to overcome this divergence is another method but worth undertaking efforts to get it to work, as its advantages of simplicity and flexibility make it an interesting method for apply in real cases.

2. METHODOLOGY

The term "*method of successive substitutions*" refers to a broad class of iterative schemes for nonlinear equations. However, the interest of this work focuses on the method of fixed point iteration.

If starting the basic equation f(x) = 0 may be written as equation (1),

$$x = g(x), \qquad (1)$$

so we could write an iterative scheme in terms of equation (2),

$$x_{i+1} = g(x_i), \qquad (2)$$

where the iteration index i=0,1,2, ... and x0 is the initial estimate of the root. This method is called fixed point iteration and its great advantage is the simplicity and the flexibility to choose the form of g(x). However, it has

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serious difficulties in cases where the iterative formula does not always converge for g(x) chosen arbitrarily. To ensure convergence of the iterative scheme for the interval containing the root, |g(x) |<1 condition must be satisfied.

Figures 1 to 4 illustrate how g'(x) affects the convergence of the method such that: if 0<g'(x) <1 is asymptotic convergence, and if 1<g'(x) <0 is oscillatory convergence. Otherwise, the method diverges. By extension of this analysis it can be demonstrated that the convergence speed increases as g'(x) approaches zero.

Resolving divergences shown in Figures 2 and 4 is interesting. For this, two strategies which are applied to a particular case are proposed. This case is the function to determine the square root of any number *t*.

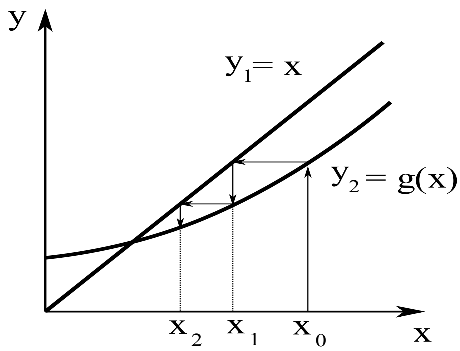

Figure-1. Convergence for 0 <g'(x)<1 (monotone behavior).

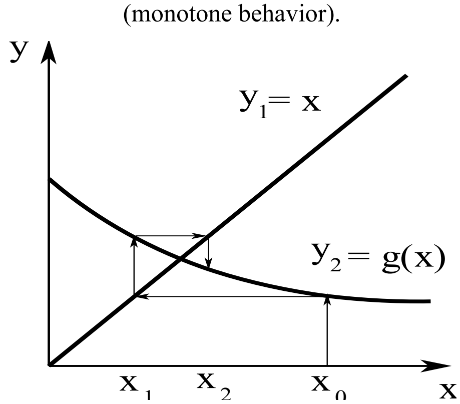

Figure-2. Convergence for -1<g'(x) <0 (oscillatory behavior).

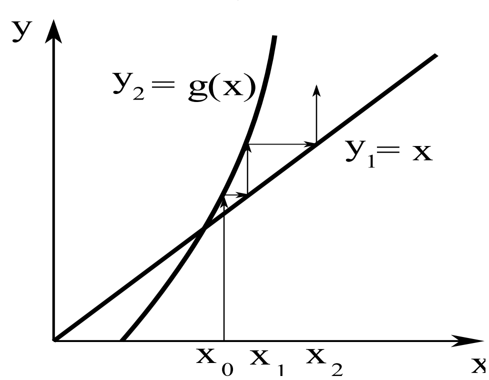

Figure-3. Divergence for g'(x)>1(monotone behavior).

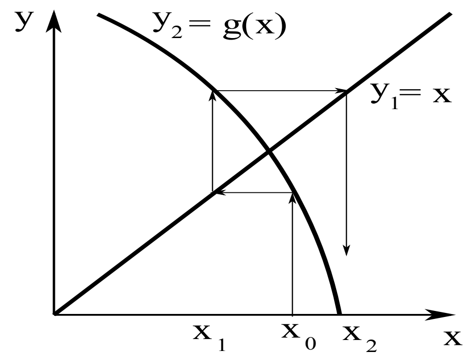

Figure-4. Convergence for g'(x) <-1 (oscillatory behavior).

2.1. Proposed strategies

2.1.1. Strategy number 1. Nonorthogonal linesearch

Based on the case of Figure-3, we propose use non-orthogonal linesearch instead of use the search orthogonal directions as in the method of fixed-point iteration. This strategy should aim to generate a convergent iterative scheme.

From Figure-5 we can observe graphically that the strategy in principle has the possibility to converge and that the proposed method can enclose to the root at least intuitively (intersection of y1 and y2) as shown.

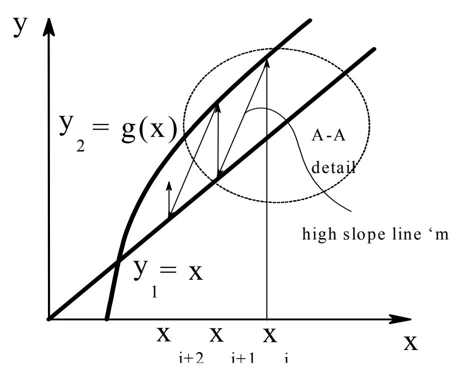

Figure-5. Convergence of nonorthogonal linesearch strategy (monotone behavior).

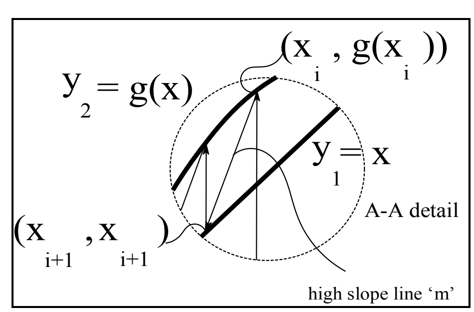

Figure-6. A-A detail.

From Figure-6 an expression for m is obtained in the form of equation (3),

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$$m = \frac{x_{i+1} - g(x_i)}{x_{i+1} - x_i}, \quad (3)$$

Starting from equation (3) is possible to get x_{i+1} by means the equation (4),

$$x_{i+1} = \frac{mx_i - g(x_i)}{m-1}. \quad (4)$$

Equation (4) represents the generalized iteration formula to overcome the problems of divergence in the cases of Figures 3 and 4, according to the following recommendations:

1) If a positive value m (Figure-3) is required, we must use m> 1, preferably beginning to test values m = 2,3, ..., If a positive value m (Figure-3) is required, use m> 1, preferably beginning to test values m = 2,3, ..., because we must search a high slope higher than the slope of y1 line (m=1) and large enough to cross the line y1 in a point before to the intersection between y1 and y2, as shown in Figure-6.

2) If a negative value m (Figure-4) is required, we must use m≤1, preferably beginning to test with values m=-1, -2, -3, ..., because a high slope must be found. In this case using linesearch with slope m=-1 is not prohibited because these would be perpendicular to y1, therefore, intersection with y1 is guaranteed.

These two recommendations have been drawn from simple approaches and have been implemented in several examples to establish its performance.

2.1.2 Strategy number 2. Rotation of reference system XY and determination of iteration formulas in the new system (Strategy of rotated axes)

In this case we take an arbitrary point (x, y) in the coordinate system XY and we seek a relationship with the coordinates of the point in a system rotated an angle θ.

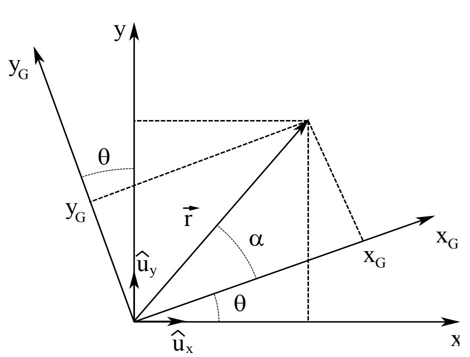

Figure-7. Transforming orthogonal coordinate system.

According to Figure-7 we can see that vector ‘r’ can be represented by equation (5).

$$\vec{r} = x_G i + y_G j, \quad (5)$$

and the unit vectors ux and uy are expressed by equations (6) and (7) respectively.

$$\hat{u}_x = (\cos\theta)\, i - (sen\theta) j \quad (6)$$
$$\hat{u}_y = (sen\,\theta)\, i + (\cos\theta) j. \quad (7)$$

Equations (8) and (9) are obtained from the projection of a vector,

$$x = \vec{r}\cdot\hat{u}_x \quad (8)$$

$$y = \vec{r}\cdot\hat{u}_y \quad (9)$$

Equations (10) and (11) are obtained operating appropriately,

$$x = (\cos\theta) x_G - (sen\theta) y_G \quad (10)$$
$$y = (sen\,\theta) x_G + (\cos\theta) y_G. \quad (11)$$

Writing equations (10) and (11) in matrix form, the system of equations (12) is obtained.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -sen\theta \\ sen\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_G \\ y_G \end{bmatrix}. \quad (12)$$

Establishing the inverse relationship in the form of equation (13) is possible because they are orthogonal systems.

$$\begin{bmatrix} x_G \\ y_G \end{bmatrix} = \begin{bmatrix} \cos\theta & sen\theta \\ -sen\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \quad (13)$$

Equations (14) and (15) are obtained when we apply an arbitrary point on the line y=x.

$$x = \frac{x_G}{\cos\theta + sen\theta}, \quad (14)$$

$$y = \frac{y_G}{\cos\theta - sen\theta}. \quad (15)$$

Knowing that equations (14) and (15) correspond to the line *y*=*x* we can establish relationships between x_G and y_G in the rotated system through matching. Equation (16) demonstrates this consideration.

$$\frac{x_G}{\cos\theta + sen\theta} = \frac{y_G}{\cos\theta - sen\theta} \quad (16)$$

Solving for *x*, equation (17) is obtained.

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$$x_G = y_G \frac{\cos\theta + sen\,\theta}{\cos\theta - sen\,\theta} \quad (17)$$

However, if the classic strategy of root finding for method of fixed point iteration is developed in the rotated system, we have $y_{Gi+1}=g(x_{Gi})$, where the superscripts indicate the iteration and the subscripts indicate the reference coordinate system. An iteration formula is obtained when we apply this reasoning to equation (17). This expression can be used in the rotated system also in the original system using equation (10).

Equation (18) shows the final iteration expression obtained in the coordinate system when is rotated an angle 'θ'.

$$x_G^{i+1} = g(x_G^i)\left[\frac{\cos\theta + sen\,\theta}{\cos\theta - sen\,\theta}\right]. \quad (18)$$

The factor in square brackets can be set as a constant called FT because the angle A is a constant. So the expression (18) can be simplified and we can get the equation (19).

$$x_G^{i+1} = FT * g(x_G^i). \quad (19)$$

The results of equation (19) can be transferred to the XY system to get the value of the root, even if g(x) does not meet the convergence condition described above; however, when we moved to the system rotated coordinate we are doing a transformation and a reformulation and this allows convergence.

3. RESULTS AND DISCUSSIONS

In order to present evidence about the performance of the proposed methods we developed a case exhibiting divergence problems with the method of fixed point iteration. This is the function that determines the square root of an arbitrary number, for example, the square root of number 2.

The starting equation is of the form described by equation (20):

$$f(x) = x^2 - 2 = 0. \quad (20)$$

Equation (21) provides g(x) in general form as g(x)=f(x)+x, so,

$$g(x) = x^2 - 2 + x. \quad (21)$$

We know in advance that the positive root of f(x) is 1,4142135623731..., however, considering $g'(x)=2x+1$ and applying the condition of convergence, we can obtain $-1<2x+1<1$. Therefore: $-1<x<0$.

This means that the scheme of fixed-point iteration is converging in an interval that does not enclose the zeros of the function. In other words, the function g(x) is divergent in the scheme of fixed-point iteration and therefore seeking a solution to the problem is necessary.

The results of the two proposed strategies are described. The implementation of the formulations was developed in a spreadsheet.

The most interesting analysis that can probably be done is the convergence curve vs. other classical methods mentioned in the introduction that can be reviewed in references (Mathews and Fink, 2011) (Burden and Faires, 2011). Figures 8 and 9 show these curves, these are used as a performance comparison of the methods.

The methods exhibit similar behavior in the convergence when we use the true error and the approximate error. The real error is calculated at an each iteration using the exact value of the root. The estimated error is calculated each iteration taking the value of the previous iteration.

Figure-8 shows how the two alternatives evaluated far exceed the speed of convergence of the methods of bisection and False Rule.

We can see that the proposed alternatives exceed the convergence of Secant method and Newton's method in the first iterations. Note these methods are known for having high speed of convergence. However, in the end, these two methods exhibit lower relative errors than the proposed strategies, although the orders of magnitude of the errors are very low. So, we can say that the proposed strategies are very competitive.

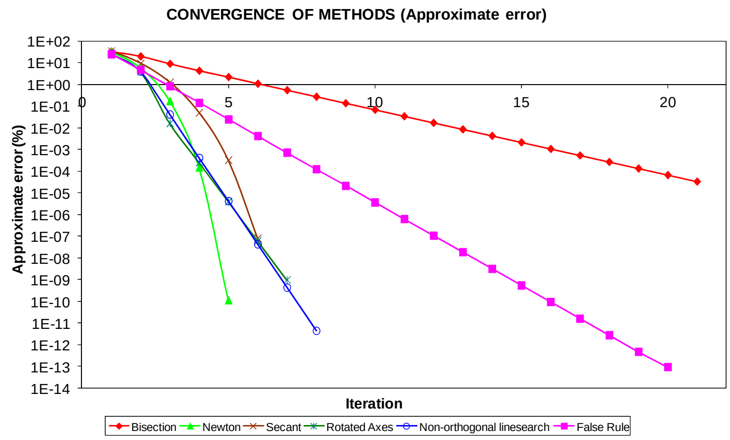

Figure-8. Comparison of convergence of the strategies evaluated with classical methods (calculations are based on the estimated error).

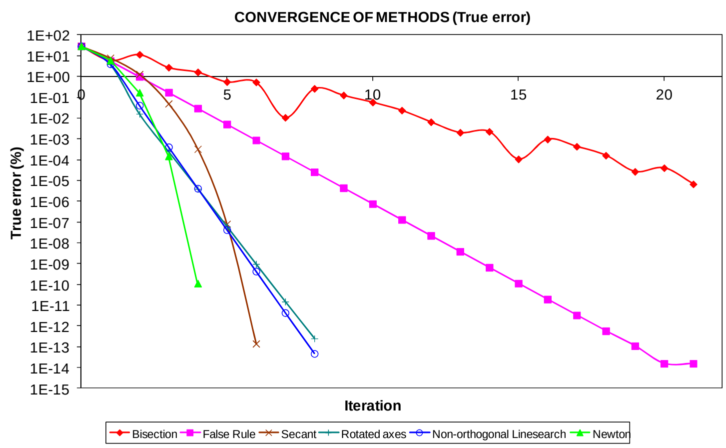

Figure-9. Comparison of convergence of the strategies evaluated with classical methods (calculations are based on the true error).

The proposed strategies are characterized by convergence that is greater than the quadratic in the first iterations then it becomes classical linear convergence of the fixed-point iteration method from they were derived. The behavior of the nonorthogonal linesearch strategy is explained by Figure-10.

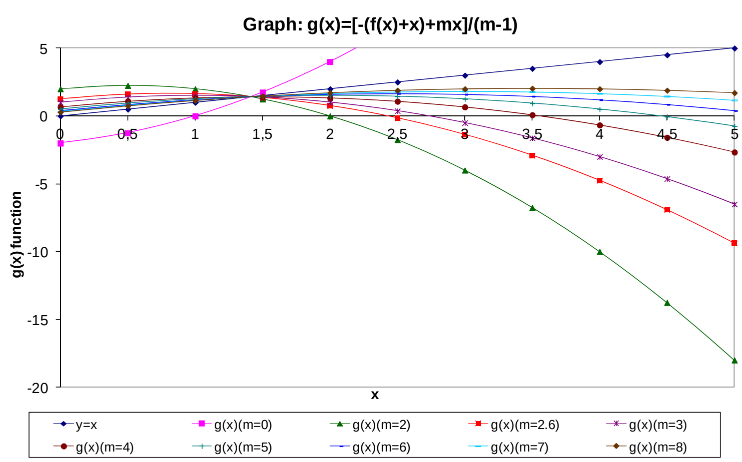

Figure-10. g(x) function using different values of parameter ‘m’. Nonorthogonal linesearch strategy.

Figure-10 indicates that the event of divergence reported in Figure-3 is presented when m=0, however, when we use m=2 the curvature changes and it becomes more flat in the vicinity of the cut with the line y=x when higher values of 'm' are used. We can say that there is an optimum value of 'm' (in present case is 3, 8). Therefore, this value is the one used for the calculations. The reason is, insofar as g(x) tends to have horizontal slope in the vicinity of the fixed point (that is, it tends to the root of f(x)) the optimum condition of iteration fixed point (g'(x) =0) is reached.

In Table-1 the results of the first four iterations of the methods are recorded.

Table-1. Results of strategies compared to the Newton method (exact value =1.4142135623731).

Iteration	**Nonorthogonal linesearch** (m=3,8)	**Rotated axes** (θ=75º)	**Newton**
0	1	1	1
1	1,357142857	1,359086014	1.5
2	1,413629738	1,413984953	1,416666667
3	1,414219368	1,414217121	1,414215686
4	1,414213503	1,414213507	1,414213562

4. CONCLUSIONS

From the results in Table-1 we can see that the proposed strategies are working properly. Just look at the insignificant differences when comparing the results with those of Newton method and remember that its quadratic convergence is the highest among open and closed classical methods.

The strategies developed show significant advantages in both implementation and precision-accuracy. Most importantly is, they solve the divergence properly and they fit to convergence rates that are very attractive for solving problems in science and engineering. An interesting advantage of the proposed search strategies is that it isn't necessary to know the derivatives of the function or enclose the desired root likewise the computational cost is certainly lower than of Newton and Secant methods.

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