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STUDY OF BUILDING AN ANALYTICAL SOLUTION OF THE AXISYMMETRIC PROBLEM OF LINEAR ELASTICITY IN STRESSES AS EXEMPLIFIED BY FINDING THE STRESS-STRAINSTATE OF AN ELLIPSOID COCAVITYUNDER THE INNER PRESSURE

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ABSTRACT

This article presents an approach to finding analytical solutions of the axisymmetric problem of linear elasticity, which is based on setting up the problem fully formulated in stresses. It closely studies the example of finding stress-strain state of an ellipsoid cocavity under the inner pressure.

Keywords: ellipsoid cocavity, analytical solution, axisymmetric problems, setting up problem, stress, cocavity, exact solution.

INTRODUCTION

Analytical methods of elastic solution has recently not decreased in relevance but became even more important, notwithstanding that for finding stress-strain state specialized software packages are widely used. It is connected with the fact that problems arising in modern technology are more complicated, especially after appearing of new materials. Without strict analytical estimations, correctness of a software-based solution can hardly be proved.

Many researchers, for example [1-4], studied the stress-strain state of Figures with stress raisers, such as pores, elastic or hard inclusions. A more detailed reference list can be found in [5]. Such inhomogeneities are seen in building and composition materials, geology, medical science [6-10]. Often these problems are axisymmetric and are usually solved via Lame's equation or Love’s function. It should be noted that when solving certain problems via specified methods one encounters difficulties in subordinating a solution to boundary conditions because of boundary values’ complexity.

As distinct from well-known approaches to solving such problems, this work uses setting up in stresses, suggested by Shamina V.A.[11].

Main part

The essence of the approach is the following:

a) Main equations are two equilibrium equations and two uniformity equations, written in stresses. Static and kinematic boundary values are written in stresses.
b) The solution is represented in the form of power series in angle’s cosine between the axis of rotation and sphere radius. These series’ coefficients, depending on radial coordinate of spherical coordinate system, are calculated via system of ordinary differential equations of Euler.

The advantage of this approach is that this system’s indeterminates coincide with static and kinematic boundary values, and this, in its turn, simplifies satisfying of boundary conditions on spherical surface.

To formulate setting up of a problem, let’s use relations given in [5].
Stress tensor is represented in the form of

$$\Sigma = \sigma_{\rho\rho}(r,\theta)\vec{e}_\rho\vec{e}_\rho + \sigma_{zz}(r,\theta)\vec{k}\vec{k} + \sigma_{\varphi\varphi}(r,\theta)\vec{e}_\varphi\vec{e}_\varphi + \sigma_{\rho z}(r,\theta)(\vec{e}_\rho\vec{k} + \vec{k}\vec{e}_\rho), \quad (1)$$

displacement vector -
$u = u_\rho(r,\theta)\vec{e}_\rho + u_z(r,\theta)\vec{k}$.

Here ρ, φ, z are cylindrical coordinates with unit vectors $\vec{e}_\rho, \vec{e}_\varphi, \vec{k}$; r, θ, φ are spherical coordinates with unit vectors $\vec{e}_1, \vec{e}_2, \vec{e}_\varphi$ (Figure-1). Axis z coincides with Figure's rotation axis. Thus, though stress tensor is written in cylindrical coordinates, but independent arguments are coordinates (r,θ). Stress tensor components (1) satisfy the following differential equations:

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$$\begin{cases} \frac{\partial \sigma_{\rho\rho}}{\partial r}\sin\theta + \frac{\partial \sigma_{\rho z}}{\partial r}\cos\theta + \frac{1}{r}\left(\frac{\partial \sigma_{\rho\rho}}{\partial \theta}\cos\theta - \frac{\partial \sigma_{\rho z}}{\partial \theta}\sin\theta\right) + \frac{\partial \sigma_3}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial \sigma_3}{\partial \theta}\cos\theta = 0, \\ r\sin\theta\left[\frac{\partial \sigma_{zz}}{\partial r}\cos\theta + \frac{\partial \sigma_{\rho z}}{\partial r}\sin\theta + \frac{1}{r}\left(\frac{\partial \sigma_{\rho z}}{\partial \theta}\cos\theta - \frac{\partial \sigma_{zz}}{\partial \theta}\sin\theta\right)\right] + \sigma_{\rho z} = 0, \\ \sigma_{\rho\rho}\frac{2\mu+\lambda}{2(\lambda+\mu)} - \sigma_{zz}\frac{\lambda}{2(\lambda+\mu)} - \sigma_3\frac{2\mu+3\lambda}{2(\lambda+\mu)} - r\sin\theta\left(\sin\theta\frac{\partial \sigma_3}{\partial r} + \frac{1}{r}\frac{\partial \sigma_3}{\partial \theta}\cos\theta\right) = 0, \\ r\sin\theta\frac{\partial}{\partial r}\left(\sigma_{\rho\rho} - \sigma_{zz}\right) + \cos\theta\frac{\partial}{\partial \theta}\left(\sigma_{\rho\rho} - \sigma_{zz}\right) + 2r\cos\theta\frac{\partial \sigma_{\rho z}}{\partial r} - 2\sin\theta\frac{\partial \sigma_{\rho z}}{\partial \theta} - \\ -\left[\sin\theta\left(r^2\frac{\partial^2 \sigma_3}{\partial r^2} + 3r\frac{\partial \sigma_3}{\partial r} + \frac{\partial^2 \sigma_3}{\partial \theta^2}\right) + 2\cos\theta\frac{\partial \sigma_3}{\partial \theta}\right] = 0. \end{cases} \quad (2)$$

where

$$\sigma_3 = 2\mu\frac{u_\rho}{\rho}, \quad \sigma_{\varphi\varphi} = \frac{3\lambda+2\mu}{2(\lambda+\mu)}\sigma_3 + \frac{\lambda}{2(\lambda+\mu)}\left(\sigma_{\rho\rho} + \sigma_{zz}\right) \quad (3)$$

λ and μ are Lame's physical constants.
Axial component of displacement is found via following equations (3):

$$\begin{cases} 2\mu\frac{\partial u_z}{\partial r} = \cos\theta(\sigma_{zz} - \sigma_{\rho\rho}) + 2\sin\theta\sigma_{\rho z} + \sin\theta\frac{\partial \sigma_3}{\partial \theta} + \sigma_3\cos\theta, \\ 2\mu\frac{1}{r}\frac{\partial u_z}{\partial \theta} = -\sin\theta(\sigma_{zz} - \sigma_{\rho\rho}) + 2\cos\theta\sigma_{\rho z} - r\sin\theta\frac{\partial \sigma_3}{\partial r} - \sigma_3\sin\theta. \end{cases} \quad (4)$$

There you could see theoretical foundations of the approach we represent. This approach helped to solve benchmark problems of stretching on the infinite distance of a taut space, which contained single inhomogeneity in the form of a spherical pore, orbicular hard or elastic inclusion, and also Lame's problem for a thickwalled spherical vessel. Obtained analytical solutions coincided with those which had been published before [12, 13].

Now we will show applicability of the method under study for problems with bodies, boundaries of which are close to spherical one.

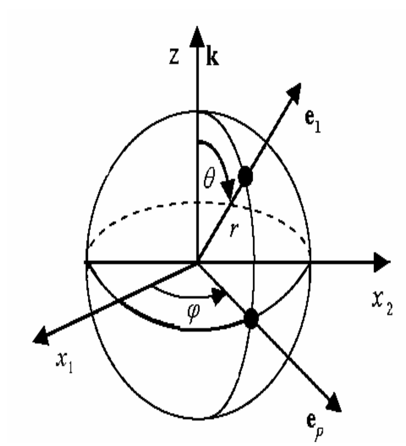

Figure-1.

Let us assume that in the space under pressure there is an inhomogeneity in the form of rotation ellipsoid with semi-axes a, b. Let us introduce coordinates (r, θ) in such a way that coordinate lines $r = const$ coincide with boundaries of a body (inhomogeneity). Then coordinates will be connected via Joukowski function:

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$$z = R\left(r + \frac{\varepsilon^2}{r}\right)\cos\theta,\ \rho = R\left(r - \frac{\varepsilon^2}{r}\right)\sin\theta,\ \varepsilon^2 = \frac{a-b}{a+b},\ R = \frac{a+b}{2}. \tag{5}$$

Substituting (5) into (2), considering that desired stresses will be sought in the form of small-parameter expansion ε:

$$\sigma_{ij} = \sum_{k=0}^{N} \varepsilon^{2k} \sigma_{ij,k}, \tag{6}$$

we will reconstitute the main system into the following one:

$$\begin{cases} \frac{\partial\sigma_{\rho\rho,k}}{\partial r}\sin\theta + \frac{\partial\sigma_{\rho z,k}}{\partial r}\cos\theta + \frac{1}{r}\left(\frac{\partial\sigma_{\rho\rho,k}}{\partial\theta}\cos\theta - \frac{\partial\sigma_{\rho z,k}}{\partial\theta}\sin\theta\right) + \frac{\partial\sigma_{3,k}}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial\sigma_{3,k}}{\partial\theta}\cos\theta + RF_\rho = \Psi_1, \\ r\sin\theta\left\{\frac{\partial\sigma_{zz,k}}{\partial r}\cos\theta + \frac{\partial\sigma_{\rho z,k}}{\partial r}\sin\theta + \frac{1}{r}\left(\frac{\partial\sigma_{\rho z,k}}{\partial\theta}\cos\theta - \frac{\partial\sigma_{zz,k}}{\partial\theta}\sin\theta\right) + \sigma_{\rho z,k} + RrF_{z,k}\sin\theta\right\} = \Psi_2 \\ \left[\sigma_{\rho\rho,k}\frac{2\mu+\lambda}{2(\lambda+\mu)} - \sigma_{zz,k}\frac{\lambda}{2(\lambda+\mu)} - \sigma_{3,k}\frac{3\lambda+2\mu}{2(\lambda+\mu)}\right] - r\sin\theta\left\{\frac{\partial\sigma_{3,k}}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial\sigma_{3,k}}{\partial\theta}\cos\theta\right\} = \Psi_5 \\ r\sin\theta\frac{\partial}{\partial r}\left(\sigma_{\rho\rho,k} - \sigma_{zz,k}\right) + \cos\theta\frac{\partial}{\partial\theta}\left(\sigma_{\rho\rho,k} - \sigma_{zz,k}\right) + 2r\cos\theta\frac{\partial\sigma_{\rho z,k}}{\partial r} - 2\sin\theta\frac{\partial\sigma_{\rho z,k}}{\partial\theta} - \\ -\left[\sin\theta\left(r^2\frac{\partial^2\sigma_{3,k}}{\partial r^2} + 3r\frac{\partial\sigma_{3,k}}{\partial r} + \frac{\partial^2\sigma_{3,k}}{\partial\theta^2}\right) + 2\cos\theta\frac{\partial\sigma_{3,k}}{\partial\theta}\right] = \Psi_3. \end{cases} \tag{7}$$

where

$$\Psi_1 = -[\left(\frac{1}{r}\right)^2[\frac{\partial\sigma_{\rho\rho,k-1}}{\partial r}\sin\theta - \frac{\partial\sigma_{\rho z,k-1}}{\partial r}\cos\theta - \frac{1}{r}\left(\frac{\partial\sigma_{\rho\rho,k-1}}{\partial\theta}\cos\theta + \frac{\partial\sigma_{\rho z,k-1}}{\partial\theta}\sin\theta\right) +$$

$$+\frac{\partial\sigma_{3,k-1}}{\partial r}\sin\theta - \frac{1}{r}\frac{\partial\sigma_{3,k-1}}{\partial\theta}\cos\theta + 2(1-2\cos^2\theta)RF_{\rho,k-1}] + \left(\frac{1}{r}\right)^4 F_{\rho,k-2}],$$

$$\Psi_2 = r\sin\theta\left(\frac{1}{r}\right)^4[-\frac{\partial\sigma_{zz,k-2}}{\partial r}\cos\theta + +\frac{\partial\sigma_{\rho z,k-2}}{\partial r}\sin\theta + \frac{1}{r}\left(-\frac{\partial\sigma_{\rho z,k-2}}{\partial\theta}\cos\theta - \frac{\partial\sigma_{zz,k-2}}{\partial\theta}\sin\theta\right) +$$

$$+\sigma_{\rho z,k-2} + RrF_{z,k-2}\sin\theta + 2(2\cos^2\theta-1)Rr\sin\theta F_{z,k-2}] + r\sin\theta\left(\frac{1}{r}\right)^2[\frac{r\partial\sigma_{zz,k-1}}{\partial r}\cos\theta + \frac{2\partial\sigma_{\rho z,k-1}}{r\partial\theta}\cos\theta + RF_{z,k-1} +$$

$$+r\sin\theta\left(\frac{1}{r}\right)^2[\frac{r\partial\sigma_{zz,k-1}}{\partial r}\cos\theta + \frac{2\partial\sigma_{\rho z,k-1}}{r\partial\theta}\cos\theta + RF_{z,k-1} + \frac{1}{r\sin\theta}2(2\cos^2\theta-1)(\sigma_{\rho z,k-1} + Rr\sin\theta F_{z,k-1})] + r\sin\theta\left(\frac{1}{r}\right)^6 RF_{z,k}$$

$$\Psi_5 = -[\left(\frac{1}{r}\right)^2[-2(2\cos^2\theta-1)\left[\sigma_{\rho\rho,k-1}\frac{2\mu+\lambda}{2(\lambda+\mu)} - \sigma_{zz,k-1}\frac{\lambda}{2(\lambda+\mu)} - \sigma_{3,k-1}\frac{3\lambda+2\mu}{2(\lambda+\mu)}\right] +$$

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$$+r\sin\theta\left\{\frac{\partial\sigma_{3,k-1}}{\partial r}\sin\theta+\frac{1}{r}\frac{\partial\sigma_{3,k-1}}{\partial\theta}\cos\theta\right\}]+\left(\frac{1}{r}\right)^4[\left[\sigma_{\rho\rho,k-2}\frac{2\mu+\lambda}{2(\lambda+\mu)}-\sigma_{zz,k-2}\frac{\lambda}{2(\lambda+\mu)}-\sigma_{3,k-2}\frac{3\lambda+2\mu}{2(\lambda+\mu)}\right]+$$

$$+r\sin\theta\left\{\frac{\partial\sigma_{3,k-2}}{\partial r}\sin\theta-\frac{1}{r}\frac{\partial\sigma_{3,k-2}}{\partial\theta}\cos\theta\right\}]],$$

$$\Psi_5=-[\left(\frac{1}{r}\right)^2[-2(2\cos^2\theta-1)\left[\sigma_{\rho\rho,k-1}\frac{2\mu+\lambda}{2(\lambda+\mu)}-\sigma_{zz,k-1}\frac{\lambda}{2(\lambda+\mu)}-\sigma_{3,k-1}\frac{3\lambda+2\mu}{2(\lambda+\mu)}\right]+$$

The system's solution will be sought in the form of series

$$\sigma_{\rho\rho,k}(r,\theta)=\sum_{n=0}^{\infty}\left[\sigma_{\rho\rho,n,k}(r)\right]\cos^{2n}\theta,\ \sigma_{zz,k}(r,\theta)=\sum_{n=0}^{\infty}\left[\sigma_{zz,n,k}(r)\right]\cos^{2n}\theta,$$

$$\sigma_{3,k}(r,\theta)=\sum_{n=0}^{\infty}\left[\sigma_{3,n,k}(r)\cos\theta\right]\cos^{2n}\theta,\ \sigma_{\rho z,k}(r,\theta)=\sin\theta\sum_{n=0}^{\infty}\left[\sigma_{\rho z,n,k}(r)\cos\theta\right]\cos^{2n}\theta. \quad (8)$$

$$F_{\rho,k}(r,\theta)=\sin\theta\sum_{n=0}^{\infty}\left[F_{\rho,n,k}(r)\right]\cos^{2n}\theta,\ F_{z,k}(r,\theta)=\sum_{n=0}^{\infty}\left[F_{z,n,k}(r)\cos\theta\right]\cos^{2n}\theta.(9)$$

Right-hand members of equations (8) will be also written in series

$$\Psi_1=\sum_{n=0}^{\infty}\left[\Psi^*_{1,n}(r)\right]\cos^{2n}\theta,\ \Psi_2=\sin\theta\cos\theta\sum_{n=0}^{\infty}\Psi^*_{2,n}(r)]\cos^{2n}\theta,$$

$$\Psi_5=\sum_{n=0}^{\infty}\Psi^*_{5,n}(r)\cos\theta]\cos^{2n}\theta,\ \Psi_3=\sin\theta\sum_{n=0}^{\infty}\left[\Psi^*_{3,n}\cos\theta\right]\cos^{2n}\theta.$$

This will allow us to write the system (8) in a simpler form, using the change of variables,

$$R_n(r)=r^2\left(\sigma^I_{\rho\rho,n,k}+\sigma^I_{\rho z,n-1,k}\right),\quad Z_n(r)=r^2(\sigma^I_{zz,n,k}+\sigma^I_{\rho z,n,k}-\sigma^I_{\rho z,n-1,k}),\ T_n(r)=r^2\sigma^I_{\rho z,n,k},$$
$$U_n(r)=r^2\left[2\sigma^I_{\rho z,n-1,k}-(\sigma^I_{zz,n,k}-\sigma^I_{\rho\rho,n,k})-\left(r\frac{d\sigma^I_{3,n,k}}{dr}+\sigma^I_{3,n,k}\right)\right],\ S_n(r)=r^2\sigma^I_{3,n,k} \quad (9)$$

These variables are connected with stresses and displacements via following relations:

$$\sigma_{\rho\rho,k}=\frac{1}{r^2}\sum_{n=0}^{\infty}(R_n(r)-T_{n-1})\cos^{2n}\theta,\quad \sigma_{zz,k}=\frac{1}{r^2}\sum_{n=0}^{\infty}(Z_n(r)-T_n+T_{n-1})\cos^{2n}\theta,$$
$$\sigma_{\rho z,k}=\frac{1}{r^2}\sin\theta\cos\theta\sum_{n=0}^{\infty}T_n\cos^{2n}\theta,\quad \sigma_{3,k}=\frac{1}{r^2}\sum_{n=0}^{\infty}(S_n)\cos^{2n}\theta,$$
$$\sigma_{\varphi\varphi,k}=\frac{1}{r^2}\sum_{n=0}^{\infty}(\frac{3\lambda+2\mu}{2(\mu+\lambda)}(R_n(r)-T_{n-1})+\frac{\lambda}{2(\mu+\lambda)}(R_n+Z_n(r)-T_n))\cos^{2n}\theta, \quad (10)$$
$$\frac{2\mu u_\rho(r,\theta)}{R}=\frac{1}{r}\sin\theta\sum_{n=0}^{\infty}S_n\cos^{2n}\theta,\quad \frac{2\mu u_z(r,\theta)}{R}=-\frac{1}{r}\sum_{n=0}^{\infty}\frac{U_n}{2n+1}\cos^{2n+1}\theta.$$

As a result we obtain the system

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$$\begin{cases} r\dfrac{dR_n}{dr}-(2n+1)R_n-Z_n-U_n+2(n+1)T_n-(2n+1)S_n=\Psi_{1,n}(r)=r^3\Psi^*_{1,n}(r),\\ r\dfrac{dZ_n}{dr}+2(n+1)(Z_{n+1}-Z_n-T_{n+1}+T_n)=\Psi_{2,n}(r)=r^2\Psi^*_{2,n}(r),\\ r\dfrac{dU_n}{dr}-U_n-(2n+1)(R_n-Z_n-T_n)+(2n+1)^2S_n-2(n+1)(2n+1)S_{n+1}=\Psi_{3,n}(r)=r^2\Psi^*_{3,n}(r),\\ r\dfrac{dS_n}{dr}=-U_n+S_n+R_n-Z_n+T_n.\\ -T_n+R_n+Z_n+S_n+\dfrac{2(\lambda+\mu)}{2\mu+\lambda}\{[U_n-R_n+(2n-1)S_n]+[R_{n-1}-U_{n-1}-Z_{n-1}-(2n-1)S_{n-1}]\}=\Psi_{5,n}(r)=r^2\Psi^*_{5,n}(r). \end{cases}, \quad (11)$$

which may be written in a simpler form:

$$\begin{cases} r\dfrac{dR_n}{dr}-2(n+1)R_n+2(n+1)T_n=-\dfrac{\lambda+2\mu}{2(\lambda+\mu)}(F_{2,n+1}-F_{1,n+1})+2(n+1)S_{n+1}-F_{3,n+1}+\Psi_{1,r,n}-\Psi_{5,r,n+1}\\ r\dfrac{dZ_n}{dr}-2(n+1)Z_n+2(n+1)T_n=2(n+1)F_{2,n+1}+\Psi_{2,n}\\ r\dfrac{dU_n}{dr}-2(n+1)U_n+(2n+1)T_n=(2n+1)\left[\dfrac{\lambda+2\mu}{2(\lambda+\mu)}(F_{2,n+1}-F_{1,n+1})+F_{3,n+1}\right]+\Psi_{3,n}(r)+(2n+1)\Psi_{5,n+1},\\ r\dfrac{dS_n}{dr}-2(n+1)S_n-T_n=\dfrac{\lambda+2\mu}{2(\lambda+\mu)}(F_{2,n+1}-F_{1,n+1})+F_{3,n+1}-2(n+1)S_{n+1}+\Psi_{5,n+1}(r)\\ r\dfrac{d}{dr}\left(r\dfrac{dT_n}{dr}\right)+4nr\dfrac{dT_n}{dr}+(4n^2-1)T_n=\Phi_n(r), n>0, \quad r\dfrac{d}{dr}\left(r\dfrac{dT_n}{dr}\right)+r\dfrac{dT_n}{dr}=\Phi_0(r), n=0. \end{cases} \quad (12)$$

where

$$F_{1,n+1}=R_{n+1}+S_{n+1},\ F_{2,n+1}=T_{n+1}-Z_{n+1},\quad F_{3,n+1}=R_{n+1}+S_{n+1}-U_{n+1}$$

$$M_1=\Psi_{1,0}(r)+\Psi_{2,0}(r),\ M_2=\Psi_{3,0}(r)-\Psi_{1,0}(r)+\Psi_{5,1}(r),$$

$$M_{1,n}=\Psi_{1,n}(r)+\Psi_{2,n}(r),\ M_{3,n}=\Psi_{1,n-1}(r)-\Psi_{2,n-1}(r)+\Psi_{3,n-1}(r)-(4n-1)\Psi_{5,n}(r)$$

$$\Phi_0(r)=-\frac{2(\lambda+\mu)}{\lambda+2\mu}\left(r\frac{d}{dr}\left(r\frac{d\Psi_{5,0}(r)}{dr}\right)-r\frac{dF_{3,n+1}}{dr}-2r\frac{d\Psi_{5,0}(r)}{dr}-r\frac{d(M_2)}{dr}\right)+r\frac{d(M_1)}{dr}+3r\frac{dF_{2,n+1}}{dr}-r\frac{dF_{1,n+1}}{dr},$$

$$\Phi_n(r)=3(2n+1)r\frac{dF_{2,n+1}}{dr}-(4n+1)r\frac{dF_{1,n+1}}{dr}+(2n-1)[(4n+3)F_{2,n+1}-(2n+1)F_{1,n+1}]+$$
$$+\frac{2(\lambda+\mu)}{\lambda+2\mu}\left\{(4n+1)r\frac{dF_{3,n+1}}{dr}+(4n^2-1)F_{3,n+1}+4n(n+1)(F_{2,n+1}-r\frac{dS_{n+1}}{dr})\right\}+\Phi_M$$

$$\Phi_M=\frac{\lambda+2\mu}{2(\lambda+\mu)}(-r\frac{dM_{1,n}}{dr}-(2n-1)M_{1,n})+((2n-1)(\psi_{1,n}-\psi_{3,n})+\psi_{5,n+1}(-4n^2+1)-2n\psi_{2,n}-$$
$$-r\frac{dM_3}{dr}+(2n-1)M_3)+r\frac{d}{dr}(r\frac{\psi_{5,n}}{dr}-2n\psi_{5,n})-2(n+1)(r\frac{\psi_{5,n}}{dr}-2n\psi_{5,n}).$$

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The system (12) simplicity lies in that it consists of Euler's equations. Besides, the system will be solved step by step, starting from the last equation and going to previous ones, finishing with calculating of the first equation. At each step there will be one unknown function.

In the approach we represent, statical boundary conditions are components of the vector $\vec{\sigma}_1$:

$$\vec{\sigma}_1 = \sigma_{1\rho}\vec{e}_\rho + \sigma_{1z}\vec{k}, \quad \sigma_{1\rho} = \sigma_{\rho\rho}(\vec{e}_\rho \cdot \vec{e}_1) + \sigma_{\rho z}(\vec{k} \cdot \vec{e}_1), \quad \sigma_{1z} = \sigma_{zz}(\vec{k} \cdot \vec{e}_1) + \sigma_{\rho z}(\vec{e}_\rho \cdot \vec{e}_1), \tag{13}$$

$$e_1 = \frac{1}{A}\left(\frac{\partial \rho}{\partial r}\vec{e}_\rho + \frac{\partial z}{\partial r}\vec{k}\right), \quad A^2 = \left(\frac{\partial \rho}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2$$

At the boundary $r = const$ as kinematic boundary values we do not use displacement vector components, but we use values σ_3, U_{1z} which are connected with it and may be written in stresses

$$\sigma_3 = \frac{2\mu u_\rho}{\rho}; \; U_{1z} = 2\mu\frac{du_z}{rd\theta} = 2\frac{1}{r}\frac{\partial \rho}{\partial \theta}\sigma_{\rho z} - \frac{\partial \rho}{\partial r}(\sigma_{zz} - \sigma_{\rho\rho}) - \frac{\partial \sigma_3 \rho}{\partial r}, \tag{14}$$

where

$$U_{1z} = 2\cos\theta\sigma_{\rho z} - \sin\theta(\sigma_{zz} - \sigma_{\rho\rho}) - \sin\theta(r\frac{\partial \sigma_3}{\partial r} + \sigma_3) - \frac{\varepsilon^2}{r^2}\left(2\cos\theta\sigma_{\rho z} + \sin\theta(\sigma_{zz} - \sigma_{\rho\rho}) + \sin\theta(r\frac{\partial \sigma_3}{\partial r} - \sigma_3)\right)$$

Thus, when solving system (12) with boundary conditions (13),(14) we can find stress-strain state of bodies, boundaries of which are close to spherical ones.

It is evident that in the ellipsoid case under study only boundary conditions (13) will be involved, and when finding zero-order approximation, system (12) will be homogeneous. Statical boundary conditions will be as follows:

$$\sigma_{1,\rho} = \sigma_{\rho\rho}\sin\theta + \sigma_{\rho z}\cos\theta = -p\sin\theta,$$

$$\sigma_{1,z} = \sigma_{\rho z}\sin\theta + \sigma_{zz}\cos\theta = -p\cos\theta$$

Thus, zero-order approximation will coincide with the known solution for a thick-walled sphere with an infinite radius and under inner hydrostatical pressure. Substituting this solution into the system we will find that:

$$\Psi_1 = 3r^{-6}\cos^2\theta\sin\theta, \; \Psi_2 = 3r^{-5}\cos\theta\sin\theta\left[1 - \cos^2\theta\right], \; \Psi_5 = 3r^{-5}\left(1 - 3\cos^2\theta + 2\cos^4\theta\right),$$

$$\Psi_3 = 3r^{-5}\sin\theta\left[3 - 2\cos^2\theta\right]$$

Now we will write the system in detail to find the initial approximation.

n=0

$$\begin{cases} r\dfrac{dR_0}{dr} - 2R_0 + 2T_0 = 9r^{-3} - \dfrac{2\mu+\lambda}{2(\lambda+\mu)}\left(T_1 - S_1 - Z_1 - R_1\right) - R_1 + U_1 + S_1, \\ r\dfrac{dZ_0}{dr} - 2Z_0 + 2T_0 = 3r^{-3} + 2(Z_1 - T_1) \\ r\dfrac{dU_0}{dr} - 2U_0 + T_0 = 18r^{-3} + \left[\dfrac{\lambda+2\mu}{2(\lambda+\mu)}(T_1 - S_1 - Z_1 - R_1) + R_1 - U_1 + S_1\right] \\ r\dfrac{dS_0}{dr} - 2S_0 - T_0 = -9r^{-3} + \dfrac{\lambda+2\mu}{2(\lambda+\mu)}\left(T_1 - S_1 - Z_1 - R_1\right) + R_1 - U_1 - S_1 \\ r\dfrac{d}{dr}(r\dfrac{d}{dr}T_0) + r\dfrac{d}{dr}T_0 = 9r^{-3}\left(\dfrac{2(\lambda+\mu)}{2\mu+\lambda} - 1\right) + r\dfrac{d}{dr}(3F_{2,1} + F_{1,1}) + \dfrac{2(\lambda+\mu)}{2\mu+\lambda}r\dfrac{d}{dr}F_{3,1}. \end{cases}$$

n=1

$$\begin{cases} r\dfrac{dR_1}{dr} - 4R_1 + 4T_1 = -3r^{-3} - \dfrac{2\mu+\lambda}{2(\lambda+\mu)}\left(T_2 - S_2 - Z_2 - R_2\right) - R_2 + U_2 + S_2, \\ r\dfrac{dZ_1}{dr} - 4Z_1 + 4T_1 = -3r^{-3} + 4(Z_2 - T_2) \\ r\dfrac{dU_1}{dr} - 4U_1 + 3T_1 = 24r^{-3} + \left[\dfrac{\lambda+2\mu}{2(\lambda+\mu)}(T_2 - S_2 - Z_2 - R_2) + R_2 - U_2 + S_2\right] \\ r\dfrac{dS_1}{dr} - 4S_1 - T_1 = 6r^{-3} + \dfrac{\lambda+2\mu}{2(\lambda+\mu)}\left(T_2 - S_2 - Z_2 - R_2\right) + R_2 - U_2 - S_2 \\ r\dfrac{d}{dr}(r\dfrac{d}{dr}T_1) + 4r\dfrac{d}{dr}T_1 + 3T_1 = 0 \end{cases}$$

n=2

$$\begin{cases} r\dfrac{dR_2}{dr} - 2R_2 + 2T_2 = \dfrac{2\mu+\lambda}{2(\lambda+\mu)}\left(T_3 - S_3 - Z_3 - R_3\right) - R_3 + U_3 + S_3, \\ r\dfrac{dZ_2}{dr} - 2Z_2 + 2T_2 = 2(Z_3 - T_3) \\ r\dfrac{dU_2}{dr} - 2U_2 + T_2 = \left[\dfrac{\lambda+2\mu}{2(\lambda+\mu)}(T_3 - S_3 - Z_3 - R_3) + R_3 - U_3 + S_3\right] \\ r\dfrac{dS_2}{dr} - 2S_2 - T_2 = \dfrac{\lambda+2\mu}{2(\lambda+\mu)}\left(T_3 - S_3 - Z_3 - R_3\right) + R_3 - U_3 - S \\ R_2 - Z_2 - U_2 - S_2 = \dfrac{2\mu+\lambda}{2(\lambda+\mu)}(T_3 - Z_3 - R_3 - S_3) + R_3 - 5S_3 - U_3. \end{cases}$$

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At once we can write that T_3, R_3, Z_3, U_3, S_3 will be equal to zero, because number of members in the series is limited. As a particular solution of the problem under study we can take the following values:

$$T_2 = Z_2 = R_2 = S_2 = U_2 = 0;$$

$$T_1 = 0,\ R_1 = Z_1 = \frac{3}{7r^3},\ U_1 = -\frac{24}{7r^3},\ S_1 = -\frac{6}{7r^3},$$

$$\left(F_{2,1} = -\frac{3}{7r^3},\ F_{2,1} = R_1 + S_1 = -\frac{3}{7r^3}, F_{3,1} = R_1 + S_1 - U_1 = \frac{21}{7r^3}\right)$$

$$T_0 = -\frac{15}{14r^3},\ R_0 = -\frac{9}{7r^3},\ Z_0 = -\frac{6}{7r^3},\ U_0 = \frac{39}{14r^3},\ S_0 = \frac{15}{14r^3}.$$

The general solution of the system will be composed of particular solution and homogeneous system solution, which will satisfy boundary conditions

$$R_0(1) = 0,\ R_1(1) = -3p, Z_0(1) = 3p,\ Z_1(1) = -3p. \quad (15)$$

These boundary conditions are obtained from the statical ones:

$$\sigma_{1,\rho} = \sigma_{\rho\rho}\sin\theta + \sigma_{\rho z}\cos\theta + \varepsilon^2(\sigma_{\rho\rho}\sin\theta - \sigma_{\rho z}\cos\theta) = -p(1+\varepsilon^2)\sin\theta$$

$$\sigma_{1z} = \sigma_{\rho z}\sin\theta + \sigma_{zz}\cos\theta + \varepsilon^2(\sigma_{\rho z}\sin\theta - \sigma_{zz}\cos\theta) = -p(1-\varepsilon^2)\cos\theta,$$

which after substituting $\sigma_{ij} = \sigma_{ij,0} + \varepsilon^2\sigma_{ij,1}$ work out to

$$\sin\theta\sigma_{\rho\rho,1} + \cos\theta\sigma_{\rho z,1} = -p\sin\theta - \sin\theta\sigma_{\rho\rho,0} + \cos\theta\sigma_{\rho z,0} = -3p\sin\theta\cos^2\theta$$

$$\sin\theta\sigma_{\rho z,1} + \cos\theta\sigma_{zz,1} = p\cos\theta - \sin\theta\sigma_{\rho z,0} + \cos\theta\sigma_{zz,0} = 3p\cos\theta - 3p\cos^3\theta.$$

For we seek the first approximation, boundary conditions will contain only members with zero and second power of ε. Eventually statical boundary conditions will assume the desired form (15) after they will have been written in terms of boundary values (9). Now we can write the system's solution:

$$T_1 = 90\frac{p(\lambda+\mu)}{(9\lambda+14\mu)r} - 30\frac{p(6\lambda+7\mu)}{r^3(9\lambda+14\mu)},\ S_1 = -9\frac{p(2\lambda+2\mu)}{r(9\lambda+14\mu)} + p\left(-3/7\frac{(-60\lambda-70\mu)}{9\lambda+14\mu} - 6/7\right)r^{-3},$$

$$R_1 = 36\frac{p(2\lambda+2\mu)}{r(9\lambda+14\mu)} - \frac{15}{7}\frac{p(48\lambda+56\mu)}{r^3(9\lambda+14\mu)} + 3/7\frac{p}{r^3},\ Z_1 = 36\frac{p(2\lambda+2\mu)}{r(9\lambda+14\mu)} - \frac{15}{7}\frac{p(48\lambda+56\mu)}{r^3(9\lambda+14\mu)} + 3/7\frac{p}{r^3},$$

$$U_1 = 27\frac{p(2\lambda+2\mu)}{r(9\lambda+14\mu)} - \frac{45}{28}\frac{p(48\lambda+56\mu)}{r^3(9\lambda+14\mu)} - 24/7\frac{p}{r^3},\ T_0 = -9/2\frac{p(5\lambda+2\mu)}{r(9\lambda+14\mu)} + \frac{45}{28}\frac{p(48\lambda+56\mu)}{(9\lambda+14\mu)r^3} - \frac{15}{14}\frac{p}{r^3},$$

$$S_0 = \frac{1}{2r}\left(-3\frac{pa^3(-5\lambda-2\mu)}{9\lambda+14\mu} - 12\frac{\lambda p}{(9\lambda+14\mu)}\right) - \frac{3}{28}\frac{p(48\lambda+56\mu)}{r^3(9\lambda+14\mu)} + \frac{15}{14}\frac{p}{r^3},$$

$$R_0 = 3/7\frac{p(48\lambda+56\mu)}{r^3(9\lambda+14\mu)} + \left(3\frac{p(-5\lambda-2\mu)}{9\lambda+14\mu} + \frac{6\lambda p}{9\lambda+14}\right)r^{-1} - \frac{9}{7}\frac{p}{r^3}$$

$$Z_0 = \frac{9}{7}\frac{p(48\lambda+56\mu)}{r^3(9\lambda+14\mu)} + \left(3\frac{p(-5\lambda-2\mu)}{9\lambda+14\mu} - 6\frac{p(2\lambda+2\mu)}{9\lambda+14\mu}\right)r^{-1} - 6/7\frac{p}{r^3},$$

$$U_0 = 1/3\left(18\frac{(2\mu+\lambda)p(2\lambda+2\mu)}{(9\lambda+14\mu)(2\lambda+2\mu)} + 9/2\frac{p(-5\lambda-2\mu)}{9\lambda+14\mu}\right)r + \frac{9}{28}\frac{p(48\lambda+56\mu)}{r^3(9\lambda+14\mu)} + \frac{39}{14}\frac{p}{r^3}.$$

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Using formulas (10) taking into account (6) we shall write $\sigma_{ij,1}$. Thus, the first approximation of the problem under study will be as follows:

$$\sigma_{\rho\rho}(r,\theta) = -\frac{\sigma_0}{r^3} + 3/2\frac{\sigma_0\cos^2(\theta)}{r^3} + \frac{\varepsilon^2\sigma_0}{r^2(9\lambda+14\mu)}(-3\frac{3\lambda+2\mu}{r} + 3/7\frac{(48\lambda+56\mu)}{r^3} + 3/7\frac{-27\lambda-42\mu}{r^3} +$$
$$+\left(1/2\frac{189\lambda+162\mu}{r} - 3/2\frac{(120\lambda+140\mu)}{r^3} - 3/2\frac{-9\lambda-14\mu}{r^3}\right)\cos^2(\theta) - \left(15\frac{(-12\lambda-14\mu)}{r^3} + 45\frac{2\lambda+2\mu}{r}\right)\cos^4(\theta)),$$

$$\sigma_{\rho z}(r,\theta) = -3/2\frac{\sigma_0\sin(\theta)\cos(\theta)}{r^3} + \frac{\varepsilon^2\sigma_0\sin(\theta)}{r^2(9\lambda+14\mu)}(\left(9/2\frac{-5\lambda-2\mu}{r} + \frac{45}{28}\frac{(48\lambda+56\mu)}{r^3} - \frac{15}{14}\frac{9\lambda+14\mu}{r^3}\right)\cos(\theta) +$$
$$+15\frac{(3(2\lambda+2\mu)r^2 - 12\lambda - 14\mu)\cos^3(\theta)}{r^3} \tag{16}$$

$$\sigma_{zz} = -\frac{3}{2}\frac{\sigma_0(-1/3+\cos^2(\theta))}{r^3} + \frac{\varepsilon^2\sigma_0}{r^2(9\lambda+14\mu)}(-\frac{9}{2}\frac{\lambda+2\mu}{r} - \frac{3}{14}(\frac{63\lambda+70\mu}{r^3}) +$$
$$+\left(-\frac{27}{2}\frac{3\lambda+2\mu}{r} + \frac{9}{14}\frac{(240\lambda+280\mu)}{r^3} + \frac{9}{14}\frac{-9\lambda-14\mu}{r^3}\right)\cos^2(\theta) + 15\frac{(3(2\lambda+2\mu)r^2 - 12\lambda - 14\mu)\cos^4(\theta)}{r^3})).$$

We obtained an analytical solution in the first approximation. In this way we can also obtain expressions for displacements by formulas (10).

To define its applicability limits is the same problem but without assumption of smallness of ε^2, and it was solved via finite element method implemented with the help of COMSOL Multiphysics; a similar approach was suggested in [14].

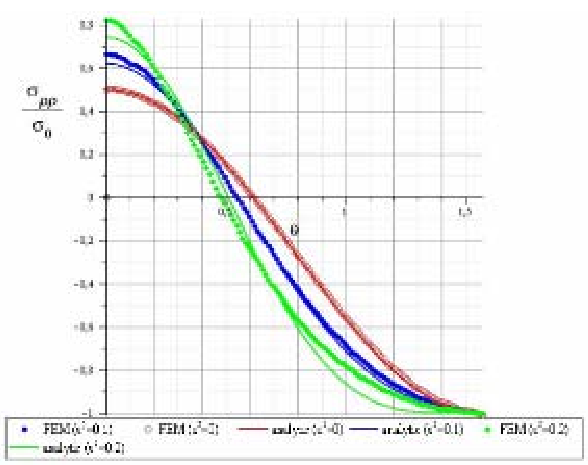

Figure-2.

From Figure-2 (stress diagrams $\sigma_{\rho\rho}/\sigma_0$ ($\lambda = 0.26$, $\mu = 0.37$) depending on the coordinate θ with different values of ε^2) it follows that analytical solution in the first approximation differs from solution obtained via FEM ($0 \le \varepsilon^2 \le 0.1$) for not more than 11%. These relations are true for all components of the stress tensor written in the cylindrical coordinates system. It should be noted that stress $\sigma_{\theta\theta}$ maximally at the poles of cocavity and under $0 \le \varepsilon^2 \le 0.04$ differs from value of this stress on the sphere for not more than 10%.

CONCLUSIONS

By the example of a problem of estimating stress-strain state of an ellipsoid cocavity under inner hydrostatical pressure, we represented detailed method of solving axisymmetric problems, setting up of which has been completely formulated in stresses. Besides, from obtained solution it follows that under $\varepsilon^2 \le 0.04$ the ellipsoid cocavity can be approximated by the sphere.

The approach we represented simplifies problem's solution subordination to boundary conditions and allows to find at the same time both stress and displacement by algebraic formulas (10).

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