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STUDY OF NONLINEAR VIBRATION OF AN ELASTICALLY RESTRAINED TAPERED BEAM USING HAMILTONIAN APPROACH

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ABSTRACT

In this paper free vibration an elastically restrained tapered beam is investigated. The governing ordinary non-linear equation of motion has been solved using Hamiltonian approach. To assess the accuracy of solutions, we compare the results with the HBM and IPM methods. The obtained results are in excellent agreement with previous results. The results show that the present method can be easily extended to other nonlinear oscillations and it can be predicted that Hamiltonian approach can be found widely applicable in engineering and physics.

Keywords: nonlinear analytical analysis; nonlinear vibration; hamiltonian approach.

1. INTRODUCTION

Many engineering structures, such as offshore structure piles, oil platform supports, oil-loading terminals, tower structures and moving arms, are modeled as tapered beams. Since these structures are usually subjected to various excitation loads, such as wind loads, wave loads and other excitations, the calculation of their nonlinear natural frequencies is required for their design recommendations [1]. The free vibration frequencies of beams have been addressed by many researchers. Timoshenko *et al*. [2] proposed the natural frequencies of a beam supported by elastic springs and Winkler foundations. The arbitrary number of spring damper masses located at intermediate abscissa has also been utilized to study the free and forced vibration of a cantilever beam. Most previous work concerned with the topic of tapered beams has studied the free vibration characteristics of tapered beams with linear and rotational restraints. The transverse vibrations of non-uniform beams with axial loads and elastically restrained ends have also been considered by Auciello [3]. The Euler-Bernoulli theory was employed by Attarnejad *et al*. [4] to obtain the exact solution for the free vibration of a tapered beam with elastic restraints. Therefore, the natural frequencies and mode shape details of an Euler- Bernoulli beam with ends on elastic supports have been derived.

On the other hand, in the last decades, scientists have proposed and applied some analytical methods to nonlinear equations. For example; the vibrational behavior of quintic nonlinear in extensional beam on two-parameter elastic substrate based on the three mode assumptions is investigated by Sedighi [5]. He employed parameter expansion method to obtain the approximate expressions of nonlinear frequency-amplitude relationship for the first, second, and third modes of vibrations. Hamiltonian approach is applied to the analysis of the nonlinear free vibration of a tapered beam by Pakar and Bayat [6]. Ghaffarzadeh and Nikkar [7] applied a new analytical method called the variational iteration method-II (VIM-II) for the differential equation of the large deformation of a cantilever beam under point load at the free tip. Askari *et al*. [8] applied He's energy balance method and He's variational approach to frequency analysis of nonlinear oscillators with rational restoring force. Sedighi and Daneshmand [9] studied nonlinear transversely vibrating beams by the homotopy perturbation method with an auxiliary term. Barari *et al*. [10] studied non-linear vibration behavior of geometrically non-linear Euler-Bernoulli beams using variational iteration method and parameter perturbation method. An efficient iterative method is applied to the analytic Treatment of Nonlinear Fifth-order Equations by Saravi and Nikkar [11]. Bagheri *et al*. [12] studied the nonlinear responses of clamped–clamped buckled beam. They used two efficient mathematical techniques called He's variational approach and Laplace iteration method in order to obtain the responses of the beam vibrations. Salehi *et al*. [13] applied two efficient methods to consider large deformation of cantilever beams under point load. Younesian *et al*. [14] studied free oscillations of beams on nonlinear elastic foundations by VIM. Askari *et al*. [15] applied higher order Hamilton approach to nonlinear vibrating systems, and many other problems solved by these methods [16-19].

Recently, the energy balance method as well as the iteration perturbation method (IPM), was utilized to analyze the non-linear problem of an elastically restrained tapered cantilever beam by Karimpour *et al*. [20]. The main goal of this paper is to present an alternative approach, namely Hamiltonian approach, for constructing highly accurate analytical approximations to the nonlinear oscillation problem.

2. MATHEMATICAL MODELLING

A schematic of the beam under study is shown in Figure-1. The physical properties, modulus of elasticity E and density ρ, of the beam are constants. The beam's thickness and width vary linearly along the beam axis. The

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restrained end of the beam is modeled by a torsional spring k_r in combination with translational spring k_t. The cross-sectional area and moment of inertia at the large end are A_1 and I_1, respectively [1].

The thickness of the beam is assumed to be small compared to the length of the beam, so that the effects of rotary inertia and shear deformation can be ignored. The beam transverse vibration can be considered to be purely planar and the amplitude of vibration may reach large values.

Let us assume a general tapered beam with a concentrated mass m on the right side, while a dashpot with damping factor c at an intermediate abscissa is placed. The left bearing is also constrained by elastic springs with two transverse and rotational stiffnesses of k_t and k_r respectively. The equation of motion by neglecting the kinetic energy of the masses is therefore suggested as [20,21]:

$$EI(z)V_1''(z,t) + \rho A(z)\ddot{V}_1(z,t) = 0; \quad 0 \le z \le z_d$$
$$EI(z)V_2''(z,t) + \rho A(z)\ddot{V}_2(z,t) = 0; \quad z_d \le z \le L \quad (1)$$

The c values as the constants of the general solutions for Eq. (1) are then found using the boundary conditions below:

$$\begin{aligned}
&v_1'''(0) + \beta v_1''(0) + K_t v_1 = 0,\\
&v_1''(0) - K_r v_1' = 0,\\
&v_1(\gamma) - v_2(\gamma) = 0,\\
&v_1'(\gamma) - v_2'(\gamma) = 0,\\
&v_1''(\gamma) - v_2''(\gamma) = 0,\\
&e^{\beta\gamma} v_1'''(\gamma) + e^{\beta\gamma} v_1''(\gamma) - e^{\beta\gamma} v_2'''(\gamma) = 0,\\
&-e^{\beta\gamma} v_2''(\gamma) - d\lambda v_1(\gamma) = 0,\\
&e^{\beta} v_2'''(1) + e^{\beta} v_2''(1) - M_z \lambda^2 v_2(1) = 0 \quad (2)
\end{aligned}$$

In Eqs. (1) and (2), z is the abscissa, L is the span of the beam, z_d is the variable location of the dashpot, ρ is the mass density, and β is a numerical exponent of the cross section. The coefficients within the above equation are defined as [20]:

$$K_R = \frac{K_r L}{EI};\ K_T = \frac{K_t L^3}{EI};\ M_z = \frac{ML^3}{EI};\ d = \frac{cL^3}{EI} \quad (3)$$

It is, however, suggested to assume the beam motion is dominated by a single active mode. Therefore, a single mode approach to discretize the continuous Lagrangian is employed. The assumption is based on the concept below, where $\varphi(z)$ and $q(t)$ are the normalized assumed mode shape of the beam and an unknown time modulation of the assumed deflection mode $\varphi_i(z)$, respectively. The beam system is eventually defined with a Lagrangian expression as described herein [20]:

$$\beta_1 = \int_0^1 A_1^* \phi^2 d\zeta, \quad (4)$$

$$\beta_2 = \int_0^1 A_1^* \left\{\int_0^{\zeta} \phi'^2 \, d\zeta\right\}^2 d\zeta, \quad (5)$$

$$\beta_3 = \int_0^1 I_1^* \phi''^2 \, d\zeta + \frac{K_t l^3}{EI_1}\phi(1)^2 + \frac{K_t l}{EI_1}\phi'(1)^2, \quad (6)$$

$$\beta_4 = \int_0^1 I_1^* \phi'^2 \phi''^2 \, d\zeta. \quad (7)$$

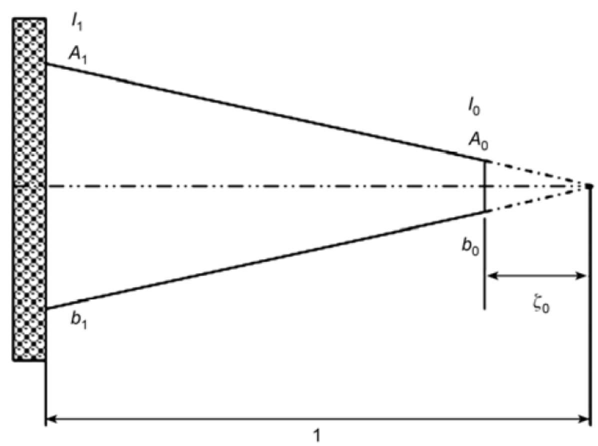

Figure-1. A schematic for the tapered beam [20].

Apply the Euler-Lagrangian relation to the system Lagrangian:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \quad (8)$$

After implementing the Euler-Lagrangian relationship, the following nonlinear dimensionless uni-modal equation of motion is obtained:

$$\ddot{q} + q + \varepsilon_1(q^2\ddot{q} + q\dot{q}^2) + \varepsilon_2 q^3 = 0 \quad (9)$$

A dot is used to denote a derivative with respect to the non-dimensional time and $t^* = (\beta^2\beta_3/\beta_1)^{0.5}t$, $\varepsilon_1 = \beta_2/\beta_1$ and $\varepsilon_2 = 2\beta_4/\beta_3$ are dimensionless coefficients.

Eq. (57) models the nonlinear, planar, flexural free vibration of the inextensible tapered beam. Indeed, the kinetic energy which arises from the inextensibility condition causes inertial nonlinearities, shown by the terms $\varepsilon_1 q\dot{q}^2$. These parameters also lead to a decrease in the natural frequency when the vibration amplitude increases.

The third term, $\varepsilon_2 q^3$ denoting the hardening static type, leads to an increase in the natural frequency when the vibration amplitude increases. As a result of the above illustrations, the behavior of the elastically restrained tapered beam analyzed through the present contribution is either hardening or softening depending on the ratio $\varepsilon_{1/}\varepsilon_2$ [22].

3. BASIC CONCEPT OF THE PROPOSED METHOD

The Previously, He [23] had introduced the energy balance method based on collocation and Hamiltonian. Recently, in 2010 it was developed into the Hamiltonian approach [24]. This approach is a kind of energy method with a vast application in conservative oscillatory systems. In order to clarify this approach, consider the following general oscillator:

$$\ddot{u} + f(u, \dot{u}, \ddot{u}) = 0 \tag{10}$$

With initial conditions:

$$u(0) = A, \quad \dot{u}(0) = 0 \tag{11}$$

Oscillatory systems contain two important physical parameters, i.e. the frequency ω and the amplitude of oscillation A. It is easy to establish a variational principle for Eq. (10), which reads;

$$J(u) = \int_0^{\frac{T}{4}} (-\frac{1}{2}\dot{u}^2 + F(u))dt \tag{12}$$

Where T is period of the nonlinear oscillator and $\dfrac{\partial F}{\partial u} = f$

In the Eq. (12), $\dot{u}^2/2$ is kinetic energy and $F(u)$ potential energy, so the Eq. (12) is the least Lagrangian action, from which we can immediately obtain its Hamiltonian, which reads:

$$H(u) = -\frac{1}{2}\dot{u}^2 + F(u) = CONSTANT \tag{13}$$

From Eq. (13), we have;

$$\frac{\partial H}{\partial A} = 0. \tag{14}$$

Introducing a new function, $\overline{H}(u)$, defined as:

$$\overline{H}(u) = \int_0^{\frac{T}{4}} (-\frac{1}{2}\dot{u}^2 + F(u))dt = \frac{1}{4}TH \tag{15}$$

Eq. (14) is, then, equivalent to the following one;

$$\frac{\partial}{\partial A}(\frac{\partial \overline{H}}{\partial T}) = 0. \tag{16}$$

Or

$$\frac{\partial}{\partial A}(\frac{\partial \overline{H}}{\partial (\frac{1}{\omega})}) = 0. \tag{17}$$

From Eq. (17) we can obtain approximate frequency-amplitude relationship of a nonlinear oscillator.

4. APPLICATION OF THE PROPOSED METHOD

Rewriting To illustrate the basic procedure of the present method, the Hamiltonian of Eq. (1) can be written in the form:

$$H(q) = -\frac{1}{2}\dot{q}^2 + \frac{1}{2}q^2 + \frac{1}{2}\varepsilon_1 q^2\dot{q}^2 + \varepsilon_2\frac{q^4}{4} \tag{18}$$

Introducing a new function, $\overline{H}(q)$. Integrating Eq. (18) with respect to t from 0 to t^*, we obtain:

$$\overline{H}(q) = \int_0^{t^*}\left\{-\frac{1}{2}\dot{q}^2 + \frac{1}{2}q^2 + \frac{1}{2}\varepsilon_1 q^2\dot{q}^2 + \varepsilon_2\frac{q^4}{4}\right\}dt^* \tag{19}$$

Assuming that the solution can be expressed as $q = A\cos(\omega t^*)$ and substituting it into Eq. (19) yields:

$$\overline{H}(A,\omega) = \int_0^{t^*}\left\{\begin{array}{l} -\frac{1}{2}A^2\omega^2\sin(\omega t^*)^2 + \frac{1}{2}A^2\cos(\omega t^*)^2 \\ +\frac{1}{2}\varepsilon_1 A^4\omega^2\sin(\omega t^*)^2\cos(\omega t^*)^2 + \frac{1}{4}\varepsilon_2 A^4\cos(\omega t^*)^4 \end{array}\right\}dt^* \tag{20}$$

The stationary condition with respect to A leads to:

$$\overline{H}(A,\omega) = \int_0^{t^*}\left\{\begin{array}{l} -A\,\omega^2\sin(\omega t^*)^2 + A\cos(\omega t^*)^2 \\ +2\varepsilon_1 A^3\omega^2\sin(\omega t^*)^2\cos(\omega t^*)^2 + \varepsilon_2 A^3\cos(\omega t^*)^4 \end{array}\right\}dt^* \tag{21}$$

Solving Eq.(21), according to ω, we have

$$\omega_{HA} = \frac{\sqrt{2}}{2}\frac{\sqrt{(1+\varepsilon_1 A^2\cos(\omega t^*)^2)(\varepsilon_2 A^2 + \varepsilon_2 A^2\cos(\omega t^*)^2 + 2)}}{(1+\varepsilon_1 A^2\cos(\omega t^*)^2)} \tag{23}$$

If we collocate $\omega t^* = \pi/4$ at we obtain:

$$\omega_{HA} = \frac{\sqrt{2}}{2}\frac{\sqrt{(2+\varepsilon_1 A^2)(4+3\varepsilon_2 A^2)}}{(1+\varepsilon_1 A^2)} \tag{24}$$

To demonstrate and verify the accuracy of this new approximate analytical approach, a comparison of the time history oscillatory displacement responses with previous results are presented in Table-1. It can be observed from Table-1 that there is an excellent agreement between the results for different values of A and constant value $\varepsilon = \varepsilon_2 = 1$

Table-1. Comparison of frequencies corresponding to various parameters of a system for different values of A.

A	$\varepsilon = \varepsilon_2 = 1$		
	ω_{HA}	ω_{HBM}	ω_{IPM}
0.1	1.00124	1.00124	1.00124
0.5	1.02740	1.02740	1.02740
1	1.08012	1.08012	1.08012
5	1.20953	1.20953	1.20953
10	1.22073	1.22073	1.22073

5. CONCLUSIONS

In this paper, the main purpose was to illustrate the application of Hamiltonian approach in solving nonlinear oscillator arising in the elastically restrained tapered beam. Also, the capabilities and facile applications of this method have been demonstrated in comparison with the harmonic balance method and iteration pertrubation method.

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