ARPN Journal of Engineering and Applied Sciences

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ANALYTICAL EXPRESSIONS FOR HEAT TRANSFER AND ENTROPY GENERATION IN A PIPE FLOW USING HOMOTOPY ANALYSIS METHOD

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ABSTRACT

This paper demonstrates the effect of convective cooling on a temperature dependent viscosity liquid flowing steadily through a cylindrical pipe. In this model, it is assumed that due to Newton's cooling law heat is exchanged with the ambient and the viscosity model varies as an inverse linear function of temperature. The analytical expressions for fluid velocity and temperature are derived using Homotopy Analysis method and entropy generation rate, total entropy generated and the Bejan number for various parametric values are determined. Our results are compared with the previous work and found to be in good agreement.

Keywords: entropy generation, pipe flow, variable viscosity, convective cooling, thermodynamic irreversibility, homotopy analysis method.

1. INTRODUCTION

There have been considerable interest in the studies related to viscous fluid with temperature dependent properties because of their application in industries like food processing, coating and polymer processing [1, 2]. The physiological fluid blood and fluids used in industries like polymer fluids have temperature dependent viscosity. Due to temperature changes these fluids alter their flow structure significantly [3-7]. Fluids with strong temperature dependence are subjected to significant changes due to viscous heating because of the coupling effect between the Navier-Stokes and energy equations. Costa and Macedonio [8] investigated the effects of viscous heat generation in fluids with temperature dependent viscosity model and concluded that these effects can play a vital role in the dynamics of magma flow. Elbashbeshy and Bazid [9] observed that the temperature dependent fluid viscosity model varies as an inverse linear function of temperature by studying the effects of temperature dependent viscosity model on heat transfer over a continuous moving surface. Makinde [10, 11] investigated the flow of a liquid flim having variable viscosity along an inclined heated plate as well as the effects temperature dependent fluid viscosity on heat transfer due to reactive flow in a cylindrical pipe.

According to second law of thermodynamics, all real processes are irreversible. Entropy generation and account of irreversibility are related in the real processes [12]. Entropy analysis can quantify thermodynamic irreversibility in a fluid flow process. Systems lose quality of energy and hence efficiency due to entropy generation [3-5, 13]. Through the books of Bejan [14, 15] studies on entropy generation in conductive and convective heat transfer processes developed significantly. Many other authors also proceeded remarkably in this study [6, 16, 17]. Recently Tshehla *et al*. [18] performed second law analysis to study the entropy generation rate in a variable viscosity liquid.

The objective of this paper is to study the effect of convective cooling on steady flow of a variable viscosity fluid through a cylindrical pipe and to investigate the entropy generation in its flow. The results are presented graphically and discussed quantitatively.

2. MATHEMATICAL FORMULATION OF THE PROBLEM

We consider the flow of a variable viscosity fluid which is in steady in the $\bar{z}$ -direction through a cylindrical pipe of radius a and length L under the action of a constant pressure gradient, viscous dissipation and convective cooling at the pipe surface [Figure-1]. The fluid is incompressible and the temperature dependent viscosity $\bar{\mu}$ can be expressed as [9].

$$\bar{\mu} = \frac{\mu_0}{l + m(\bar{T} - T_a)}, \tag{1}$$

where μ_0 is the fluid dynamic viscosity at the ambient temperature T_a

The continuity, momentum and energy equations governing the problem in dimensionless form are given by [2, 10, 18]

$$\frac{\partial}{\partial z}(ru) + \frac{\partial}{\partial r}(rv) = 0 \tag{2}$$

$$\varepsilon.Re\left(u\frac{\partial u}{\partial z} + v\frac{\partial u}{\partial r}\right) = -\frac{\partial p}{\partial z} + 2\varepsilon^2\frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left[r\mu\left(\frac{\partial u}{\partial r} + \varepsilon^2\frac{\partial v}{\partial z}\right)\right] \tag{3}$$

VOL. 10, NO. 2, FEBRUARY 2015 ISSN 1819-6608
ARPN Journal of Engineering and Applied Sciences

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$$\varepsilon^3 Re\left(u\frac{\partial v}{\partial z}+v\frac{\partial v}{\partial r}\right)=-\frac{\partial p}{\partial r}+\frac{2\varepsilon^2}{r}\frac{\partial}{\partial r}\left(r\mu\frac{\partial v}{\partial r}\right)+\varepsilon^2\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial r}+\varepsilon^2\frac{\partial v}{\partial z}\right)\right]-2\mu\varepsilon^2\frac{v}{r^2} \tag{4}$$

$$\varepsilon.RePr\left(u\frac{\partial T}{\partial z}+v\frac{\partial T}{\partial r}\right)=\varepsilon^2\frac{\partial^2 T}{\partial z^2}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\mu\Phi \tag{5}$$

where,

$$\Phi = Br\left[2\varepsilon^2\left(\frac{\partial u}{\partial z}\right)^2+2\varepsilon^2\left(\frac{\partial v}{\partial r}\right)^2+\left(\frac{\partial u}{\partial r}\right)^2+2\varepsilon^2\left(\frac{v}{r}\right)^2+2\varepsilon^2\frac{\partial v}{\partial z}\frac{\partial u}{\partial r}+\varepsilon^4\left(\frac{\partial v}{\partial z}\right)^2\right] \tag{6}$$

We have to use the following dimensionless variables from the equations (2)-(6)

$$r=\frac{\bar{r}}{\varepsilon L},z=\frac{\bar{z}}{L},u=\frac{\bar{u}}{U},v=\frac{\bar{v}}{\varepsilon U},\varepsilon=\frac{a}{L},\mu=\frac{\bar{\mu}}{\mu_0},T=\frac{\bar{T}-T_a}{T_a},P=\frac{a^2\bar{P}}{\mu_0 UL} \tag{7}$$

$$\alpha = mT_a, Br=\frac{\mu_0 U^2}{kT_a}, Pr=\frac{\mu_0 c_p}{k}, Re=\frac{\rho U a}{\mu_0}, Bi=\frac{ah}{k} \tag{8}$$

Since the aspect ratio 0 <ε<<1, from the equations (2) – (6) we obtain the following asymptotic simplification

$$0=-\frac{\partial p}{\partial z}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\mu\frac{\partial u}{\partial r}\right)+O(\varepsilon) \tag{9}$$

$$0=\frac{\partial p}{\partial r}+O(\varepsilon^2) \tag{10}$$

$$0=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\mu Br\left(\frac{\partial u}{\partial r}\right)^2+O(\varepsilon) \tag{11}$$

where $\mu = 1/(1+\alpha T)$ (12)

Now the eqns. (8) - (10) subject to the boundary conditions equations (11) and (12) can be combined and we obtain the dimensionless equations as follows:

$$\frac{du}{dr}=\frac{-rG}{2}(1+\alpha T) \tag{13}$$

and

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right)+\frac{r^3G^2Br}{4}(1+\alpha T)=0 \tag{14}$$

The corresponding boundary conditions are as follows:

$$u=0,\frac{dT}{dr}=-BiT \text{ at } r=1 \tag{15}$$

$$\frac{du}{dr}=\frac{dT}{dr}=0 \text{ at } r=0 \tag{16}$$

where $G=-\frac{\partial P}{\partial z}$ (17)

is the constant axial pressure gradient.
The entropy generation number [1] is given by

$$N_S=\frac{a^2S^m}{k}=\left(\frac{\partial T}{\partial r}\right)^2+\mu Br\left(\frac{\partial u}{\partial r}\right)^2+O(\varepsilon^2) \tag{18}$$

The dimensionless form of the total entropy generated [18] is given by

$$N_T=\frac{S^T}{2\pi Lk}=\int_0^1 N_S r dr \tag{19}$$

where N_1 and N_2 are given by

$$N_1=\left(\frac{\partial T}{\partial r}\right)^2,\ N_2=\mu Br\left(\frac{\partial u}{\partial r}\right)^2 \tag{20}$$

The Bejan number Be is the ratio of the heat transfer entropy N_1 to the overall entropy generation rate

N_S

i.e. $Be=\frac{N_1}{Ns}=\frac{1}{1+\Phi}$ (21)

3. SOLUTION OF THE PROBLEM USING HOMOTOPY ANALYSIS METHOD (HAM)

Homotopy analysis method is a non perturbative analytical method for obtaining series solutions to nonlinear equations and has been successfully applied to

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numerous problems in science and engineering [19-32]. In comparison with other perturbative and non-perturbative analytical methods, HAM offers the ability to adjust and control the convergence of a solution via the so-called convergence-control parameter. Because of this, HAM has proved to be the most effective method for obtaining analytical solutions to highly nonlinear differential equations. Previous applications of HAM have mainly focused on nonlinear differential equations in which the non-linearity is a polynomial in terms of the unknown function and its derivatives.

Liao [19-27] proposed this powerful analytical method for nonlinear problems, namely the Homotopy analysis method. This method offers an analytical solution in terms of an infinite power series. Nevertheless, on that point is a pragmatic need to value this solution and to obtain numerical values from the infinite power series. In order to investigate the accuracy of the Homotopy analysis method (HAM) solution in a finite number of terms, the system of differential equations was solved. The Homotopy analysis method is a good technique comparing to other perturbation methods.

Homotopy perturbation method is a special instance of the Homotopy analysis method. Different from all reported perturbation and non-perturbative techniques, the Homotopy analysis method itself provides us with a convenient means to hold and adjust the convergence region and rate of approximation series, when necessary. Briefly speaking, the Homotopy analysis method has the following advantages. It is valid even if a given nonlinear problem does not hold in any small/large parameter at all it can be used to efficiently approximate a non-linear problem by selecting different sets of basis functions. The Homotopy analysis method contains the auxiliary parameter, which provides us with a simple means to adjust and hold the overlap area of the solution series. The approximate analytical expressions of the velocity field and the temperature field using the Homotopy analysis method from equations (13) and (14) with the boundary conditions (15) and (16) are as follows:

$$u(r)=\frac{G}{4}\left(1-r^2\right)+h\left\{\frac{G\alpha}{576Bi}\left[\begin{array}{c}12BrG^2r^2+3BrBiG^2r^2-BrBiG^2r^6\\-12BrG^2-2BrBiG^2\end{array}\right]\right\} \tag{22}$$

and

$$T(r)=\frac{1}{48Bi}\left[-BrBiG^2r^4+4BrG^2+BrBiG^2\right]-h\left\{\begin{array}{l}\dfrac{BrG^2}{144}r^4+\dfrac{BrG^2\alpha}{192Bi}\left(\begin{array}{c}\dfrac{BrBiG^2r^8}{56}-\\\left(4BrG^2+BrBiG^2\right)\dfrac{r^4}{12}\end{array}\right)\\-\dfrac{1}{Bi}\left[\begin{array}{l}\dfrac{BrG^2}{36}-\dfrac{BrG^2\alpha}{48Bi}\left(\dfrac{BrBiG^2}{21}+\dfrac{BrG^2}{3}\right)\\+\dfrac{BrBiG^2}{144}-\dfrac{BrG^2\alpha}{192}\left(\begin{array}{c}\dfrac{BrG^2}{3}+\\\dfrac{11}{168}BrBiG^2\end{array}\right)\end{array}\right]\end{array}\right\} \tag{23}$$

3.1. Previous work

The analytical expressions of the velocity field and the temperature field [1] from the equations (13) and (14) with the boundary conditions equations (15) and (16) are as follows:

$$u(r)=\frac{\frac{1}{4}\left(GBi\left(\begin{array}{l}-2r^2BesselJ\left(0,\frac{1}{4}G\sqrt{\alpha Br}\,r^2\right)-r^2\pi StruveH\left(0,\frac{1}{4}G\sqrt{\alpha Br}\,r^2\right)\\BesselJ\left(1,\frac{1}{4}G\sqrt{\alpha Br}\,r^2\right)+\\r^2\pi StruveH\left(1,\frac{1}{4}G\sqrt{\alpha Br}\,r^2\right)BesselJ\left(0,\frac{1}{4}G\sqrt{\alpha Br}\,r^2\right)\\+2BesselJ\left(0,\frac{1}{4}G\sqrt{\alpha Br}\right)\\+\pi StruveH\left(0,\frac{1}{4}G\sqrt{\alpha Br}\right)BesselJ\left(1,\frac{1}{4}G\sqrt{\alpha Br}\right)\\-\pi StruveH\left(1,\frac{1}{4}G\sqrt{\alpha Br}\right)BesselJ\left(0,\frac{1}{4}G\sqrt{\alpha Br}\right)\end{array}\right)\right)}{\left(\left(-\sqrt{\alpha Br}\,GBesselJ\left(1,\frac{1}{4}G\sqrt{\alpha Br}\right)\right)+2Bi\,BesselJ\left(0,\frac{1}{4}G\sqrt{\alpha Br}\right)\right)} \tag{24}$$

and

VOL. 10, NO. 2, FEBRUARY 2015 ISSN 1819-6608
ARPN Journal of Engineering and Applied Sciences

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$$T(r)=\frac{\left(2BesselJ\left(0,\frac{1}{4}G\sqrt{Br\alpha}r^2\right)Bi\right)}{\left(\left(-\sqrt{Br\alpha}GBesselJ\left(1,\frac{1}{4}G\sqrt{Br\alpha}\right)+2BiBesselJ\left(0,\frac{1}{4}G\sqrt{Br\alpha}\right)\right)\alpha\right)-\frac{1}{\alpha}} \tag{25}$$

4. RESULTS AND DISCUSSIONS

In this section we discuss the effect of the velocity $u(r)$, temperature $T(r)$, the entropy generation rate N_S, total entropy generation N_T, and the Bejan number Be with respect to the changing values of the parameters α, Br, Bi and the pressure gradient G.

Figures 2 (a)-(d) represent the dimensionless velocity profile for dimensionless radial distance r for different parameters. From these Figures we observe that the velocity increases with α, increases with Br, increases with G and increases with decreasing values of Bi. From the Figures 3 (a)-(d) we infer that the temperature with respect to the radial distance increases with α, increases with Br, increases with G and increases with the decreasing values of the Biot number Bi. Figures 4 (a)-(c) reveal that the entropy generation rate N_S for the radial distance r increases with α, increases with Br and increases with decreasing values of Bi. Figures 5 (a)-(c) indicate that the total entropy generation N_T with respect to the Brinkman number Br increases with α, increases with the pressure gradient G and increases with the decreasing values of Bi. Figures 6 (a)-(c) represent the Bejan number Be versus the radial distance r. From these Figures we observe that the Bejan number increases with α, increases with Br and increases with the decreasing values of Bi. The convective cooling in the flow system enhances when Bi increases. From the above results, we note that all the values $u(r)$, $T(r)$, N_S, N_T and Be increase when the Biot number Bi decreases.

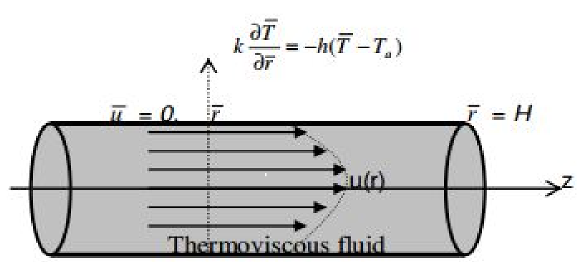

Figure-1. Schematic diagram of the problem.

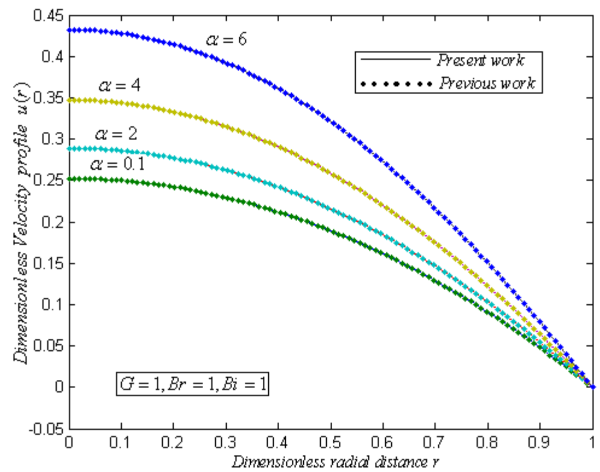

Figure-2(a). The dimensionless velocities are computed with respect to the radial distances using the equation (22) for the given fixed values of G, Br, Bi and different values of α, where $h=-0.116$.

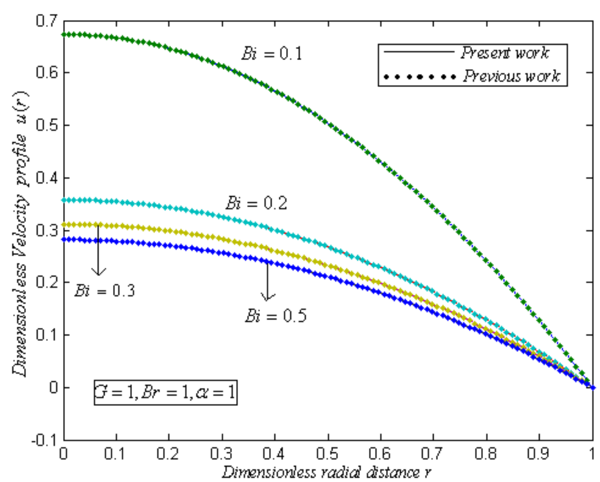

Figure-2(b). The dimensionless velocities are computed with respect to the radial distances using the equation (22) for the given fixed values of G, Br, α and different values of Bi, where $h=-0.116$.

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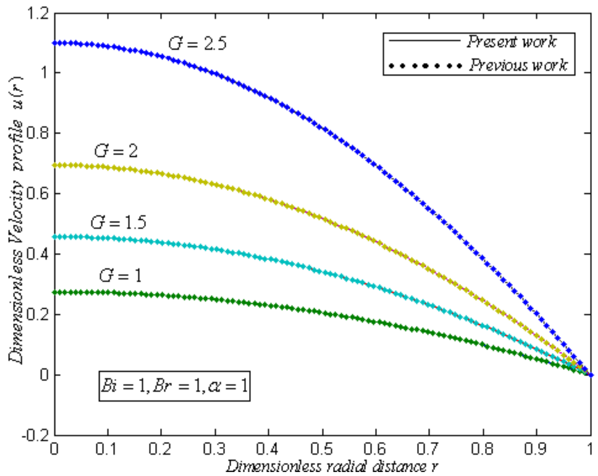

Figure-2(c). The dimensionless velocities are computed with respect to the radial distances using the equation (22) for the given fixed values of Bi, Br, α and different values of G, where $h = -0.116$.

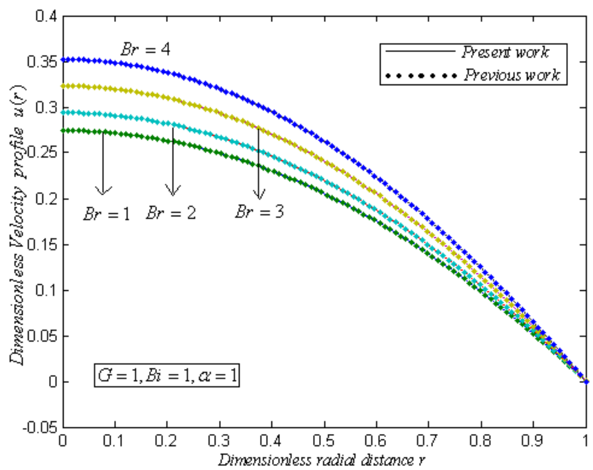

Figure-2(d). The dimensionless velocities are computed with respect to the radial distances using the equation (22) for the given fixed values of G, Bi, α and different values of Br, where $h = -0.116$.

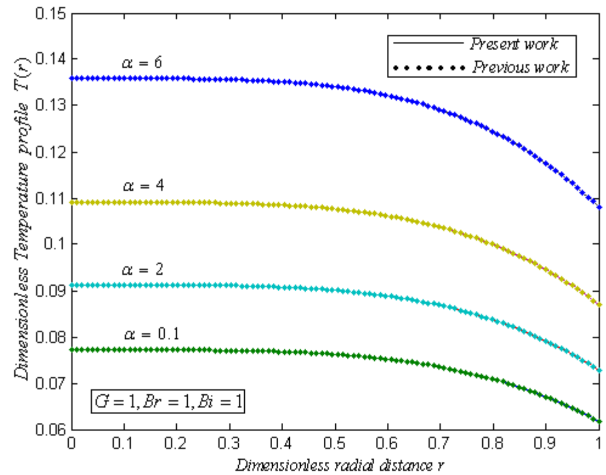

Figure-3(a). The dimensionless temperatures are computed with respect to the radial distances using the equation (23) for the given fixed values of G, Br, Bi and different values of α, where $h = -0.116$.

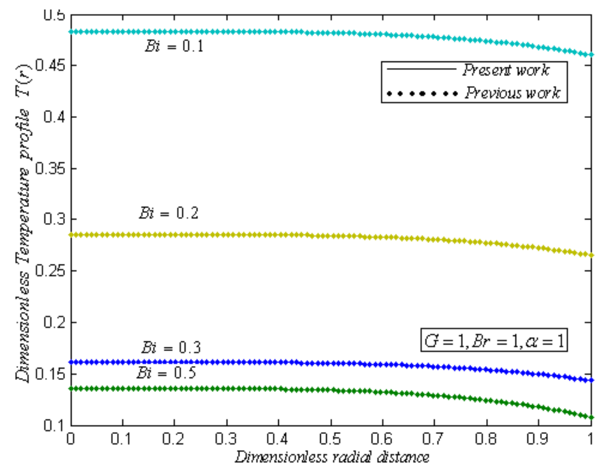

Figure-3(b). The dimensionless temperatures are computed with respect to the radial distances using the equation (23) for the given fixed values of G, Br, α and different values of Bi, where $h = -1.078$.

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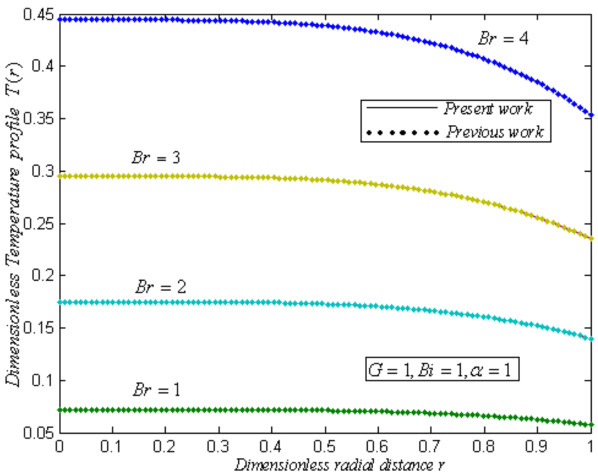

Figure-3(c). The dimensionless temperatures are computed with respect to the radial distances using the equation (23) for the given fixed values of G, Bi, α and different values of Br, where $h = -1.078$.

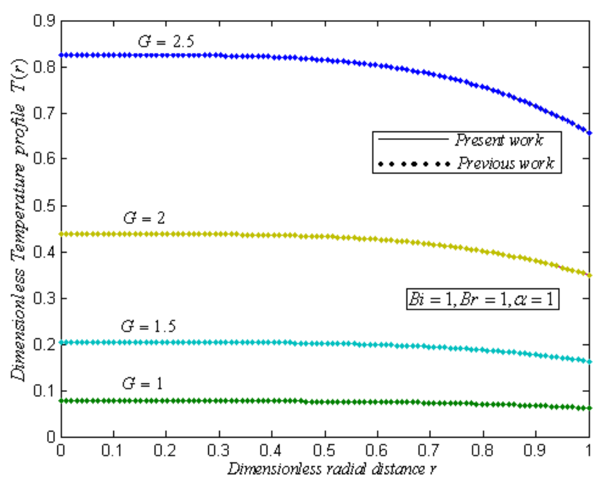

Figure-3(d). The dimensionless temperatures are computed with respect to the radial distances using the equation (23) for the given fixed values of Bi, Br, α and different values of G, where $h = -1.078$.

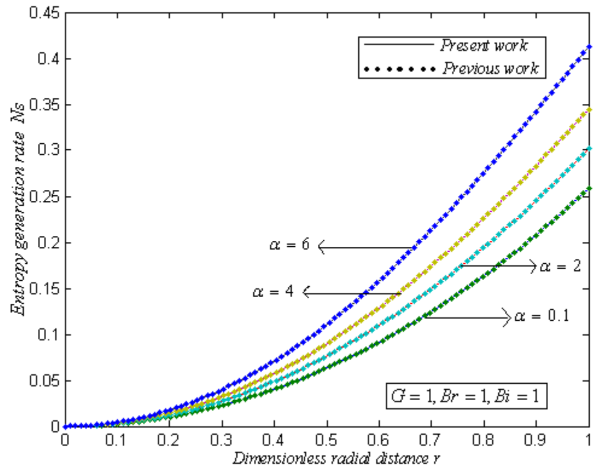

Figure-4(a). The entropy generation rates are computed with respect to the radial distances using the equation (18), (22) and (23) for the given fixed values of G, Br, Bi and different values of α, where $h = -0.102$.

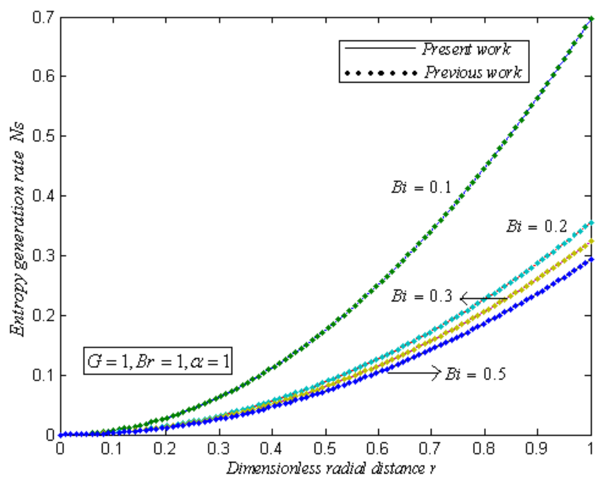

Figure-4(b). The entropy generation rates are computed with respect to the radial distances using the equation (18), (22) and (23) for the given fixed values of G, Br, α and different values of Bi, where $h = -0.102$.

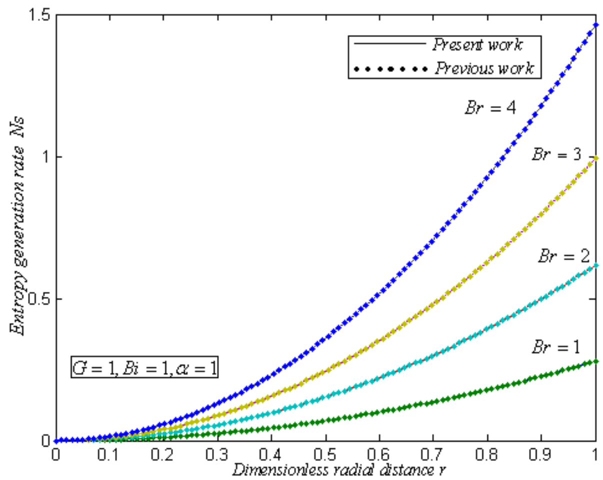

Figure-4(c). The entropy generation rates are computed with respect to the radial distances using the equation (18), (22) and (23) for the given fixed values of G, Bi, α and different values of Br, where $h = -0.102$.

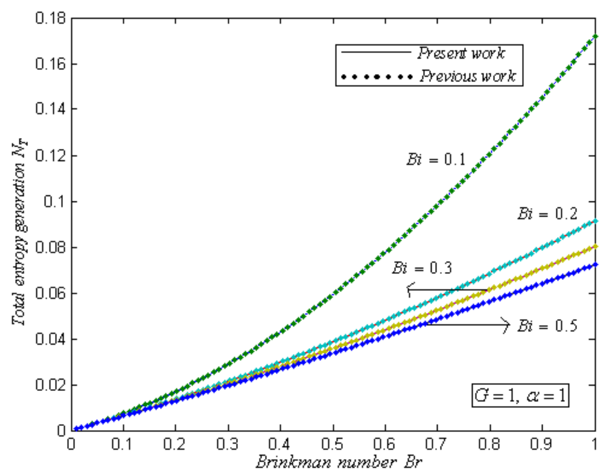

Figure-5(a). The total entropy generations N_T are computed with respect to the Brinkman number Br using the equation (18), (22) and (23) for the given fixed values of G, α and different values of Bi, where $h = -0.092$.

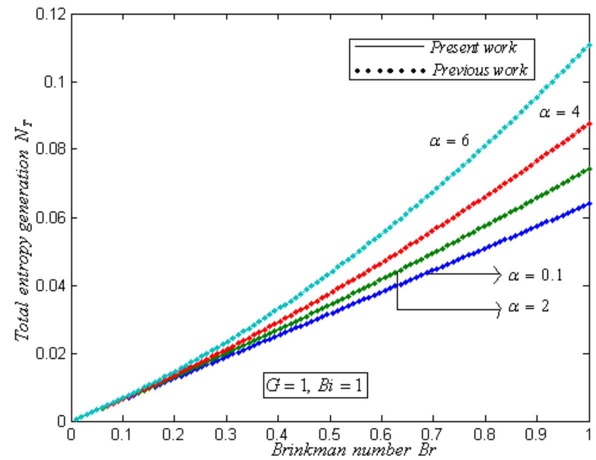

Figure-5(b). The total entropy generations N_T are computed with respect to the Brinkman number Br using the equation (19), (22) and (23) for the given fixed values of G, Bi and different values of α, where $h = -0.092$.

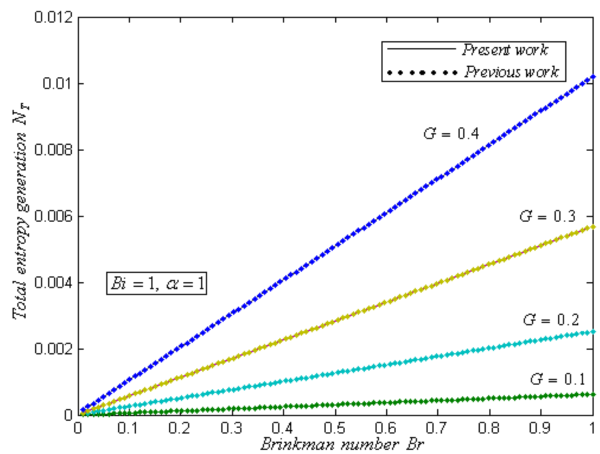

Figure-5(c). The total entropy generations N_T are computed with respect to the Brinkman number Br using the equation (19), (22) and (23) for the given fixed values of Bi, α and different values of G, where $h = -0.092$.

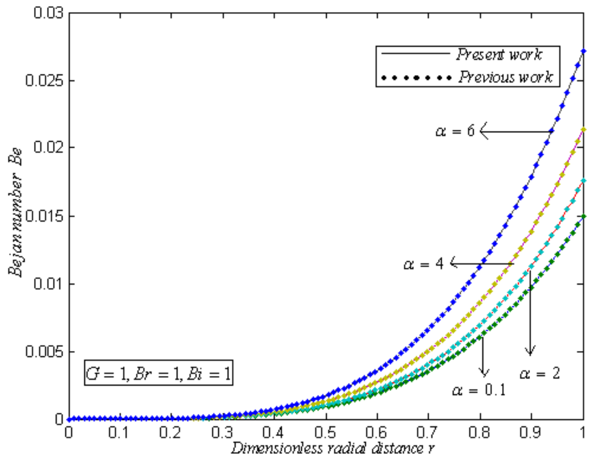

Figure-6 (a). The Bejan numbers Be are computed with respect to the dimensionless radial distance r using the equation (21)-(23) for the given fixed values of G, Br, Bi and different values of α, where $h = -0.123$.

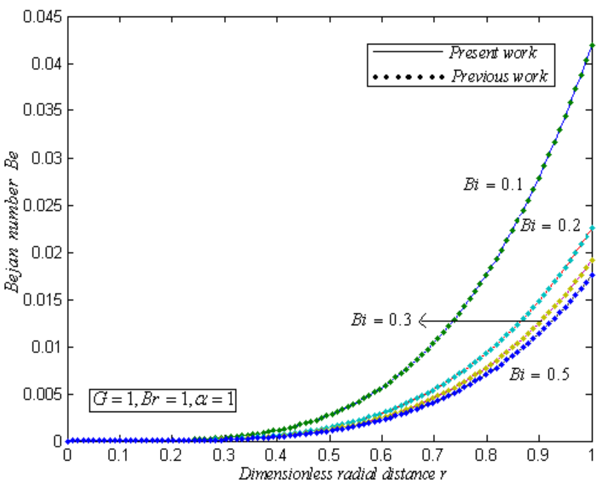

Figure-6(b). The Bejan numbers Be are computed with respect to the dimensionless radial distance r using the equation (21)-(23) for the given fixed values of G, Br, α and different values of Bi, where $h = -0.123$.

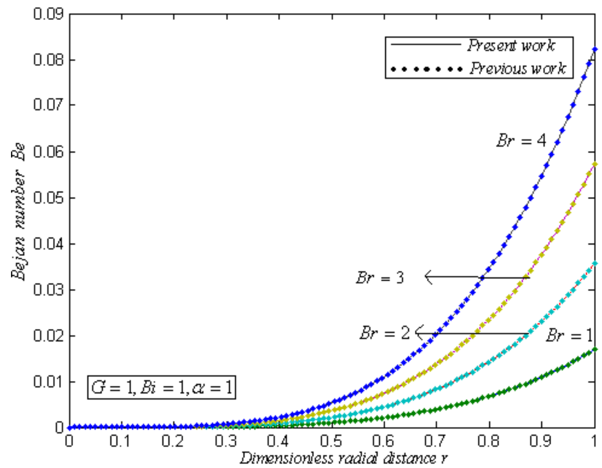

Figure-6(c). The Bejan numbers Be are computed with respect to the dimensionless radial i distance r using the equation (21)-(23) for the given fixed values of G, Bi, α and different values of Br, where $h = -0.123$.

5. CONCLUSIONS

In this paper the effect of convective cooling on a temperature dependent viscosity liquid flowing steadily through a cylindrical pipe was investigated. The velocity and temperature profiles were obtained by the Homotopy analysis method and using them the entropy generations rate, the total entropy generation and the Bejan number was determined. The results were compared with the previous work and found to be in good agreement. The Homotopy analysis method is a simple and promising method to solve various strongly non-linear differential equations in the areas of physical, chemical and biological sciences.

ACKNOWLEDGEMENT

The authors are thankful to Shri. S. Natanagopal, Secretary, The Madura College Board, Dr. R. Murali, The Principal and Mr. S. Muthukumar, Head of the Department of Mathematics, The Madura College (Autonomous), Madurai, Tamil Nadu, India for their constant encouragement.

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Appendix-A

Basic concept of the Homotopy analysis method

Consider the following differential equations

$$N[u(t)] = 0 \tag{A.1}$$

where N is a nonlinear operator, t denotes an independent variable, $u(t)$ is an unknown function. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the conventional Homotopy method, Liao [19-27] constructed the so-called zero-order deformation equation as:

$$(1-p)L[\varphi(t;p)-u_0(t)] = phH(t)N[\varphi(t;p)] \tag{A.2}$$

where $p\in[0,1]$ is the embedding parameter, $h\neq 0$ is a nonzero auxiliary parameter, $H(t)\neq 0$ is an auxiliary function, L an auxiliary linear operator, $u_0(t)$ is an initial guess of $u(t)$, $\varphi(t:p)$ is an unknown function. It is important, that one has great freedom to choose auxiliary unknowns in HAM. Obviously, when $p=0$ and $p=1$, it holds:

$$\varphi(t;0) = u_0(t) \text{ and } \varphi(t;1) = u(t) \tag{A.3}$$

respectively. Thus, as p increases from 0 to 1, the solution $\varphi(t;p)$ varies from the initial guess $u_0(t)$ to the solution $u(t)$. Expanding $\varphi(t;p)$ in Taylor series with respect to p, we have:

$$\varphi(t;p) = u_0(t) + \sum_{m=1}^{+\infty} u_m(t)p^m \tag{A.4}$$

where

$$u_m(t) = \frac{1}{m!}\frac{\partial^m \varphi(t;p)}{\partial p^m}\Big|_{p=0} \tag{A.5}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter h, and the auxiliary function are so properly chosen, the series eqn.(A.4) converges at $p=1$ then we have:

$$u(t) = u_0(t) + \sum_{m=1}^{+\infty} u_m(t). \tag{A.6}$$

Differentiating

(A.2) for m times with respect to the embedding parameter p, and then setting $p=0$ and finally dividing them by $m!$, we will have the so-called $m-th$ order deformation equation as:

$$L[u_m - \chi_m u_{m-1}] = hH(t)\Re_m(\vec{u}_{m-1}) \tag{A.7}$$

where

$$\Re_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N[\varphi(t;p)]}{\partial p^{m-1}} \tag{A.8}$$

and

$$\chi_m = \begin{cases} 0, & m\le 1, \\ 1, & m>1. \end{cases} \tag{A.9}$$

Applying L^{-1} on both side of the equation (A.7), we get

$$u_m(t) = \chi_m u_{m-1}(t) + hL^{-1}[H(t)\Re_m(\vec{u}_{m-1})] \tag{A.10}$$

In this way, it is easily to obtain u_m for $m\ge 1$, at M^{th} order, we have

$$u(t) = \sum_{m=0}^{M} u_m(t) \tag{A.11}$$

When $M\to+\infty$, we get an accurate approximation of the original equation (A.1). For the convergence of the above method we refer the reader to Liao [13]. If the equation (A.1) admits unique solution, then this method will produce the unique solution.

Appendix-B

Analytical expressions of the equations (13) - (16) using the Homotopy analysis method

In this appendix we derive the analytical expressions for $u(r)$ and $T(r)$ using the HAM

From equation (13) and (14) we get the following:

$$\frac{du}{dr} + \frac{rG}{2}(1+\alpha T) = 0 \tag{B.1}$$

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and

$$\frac{d^2T}{dr^2}+\frac{1}{r}\frac{dT}{dr}+\frac{Br}{4}G^2r^2(1+\alpha T)=0 \tag{B.2}$$

We construct the Homotopy for the above equations as follows.

$$(1-p)\left[\frac{du}{dr}+\frac{rG}{2}\right]=hp\left[\frac{du}{dr}+\frac{rG}{2}+\frac{rG}{2}\alpha T\right] \tag{B.3}$$

and

$$(1-p)\left[\frac{d^2T}{dr^2}+\frac{Br}{4}G^2r^2\right]=hp\left[\frac{d^2T}{dr^2}+\frac{1}{r}\frac{dT}{dr}+\frac{Br}{4}G^2r^2+\frac{Br}{4}G^2\alpha Tr^2\right] \tag{B.4}$$

The approximate solutions for (B.3) and (B.4) are given by

$$u=u_0+pu_1+p^2u_2+.......... \tag{B.5}$$

$$T=T_0+pT_1+p^2T_2+.......... \tag{B.6}$$

The initial approximations are as follows:

$$u_0'(0)=0;\ u_j'(0)=0 \text{ and}$$
$$u_0(1)=0; u_j(1)=0,\quad j=1,2,3,...... \tag{B.7}$$

$$T_0'(0)=0;\ T_j'(0)=0;\quad \text{and}$$
$$T_0'(1)=-BiT_0(1);\ T_j'(1)=-BiT_j(1),\ j=1,2,3...... \tag{B8}$$

Substituting the equation (B.5) into (B.3), we get

$$(1-p)\left[\frac{d}{dr}\left(u_0+pu_1+p^2u_2+\ldots\right)+\frac{rG}{2}\right]=hp\left[\frac{d}{dr}\left(u_0+pu_1+p^2u_2+\ldots\right)+\frac{rG}{2}+\frac{rG}{2}\alpha T\right] \tag{B.9}$$

Substituting the equation (B.6) into (B.4), we get

$$(1-p)\left[\frac{d^2}{dr^2}\left(T_0+pT_1+p^2T_2+..\right)+\frac{Br}{4}G^2r^2\right]=$$
$$hp\left[\begin{array}{l}\frac{d^2}{dr^2}\left(T_0+pT_1+p^2T_2+...\right)+\frac{1}{r}\frac{d}{dr}\left(T_0+pT_1+p^2T_2+...\right)\\ +\frac{Br}{4}G^2r^2+\frac{Br}{4}G^2\alpha r^2\left(T_0+pT_1+...\right)\end{array}\right] \tag{B.10}$$

Comparing the coefficients of the like powers of p in (B.9), we get

$$p^0:\frac{du_0}{dr}+\frac{rG}{2}=0 \tag{B.11}$$

$$p^1:\ \frac{du_1}{dr}-(1+h)\frac{du_0}{dr}-(1+h)\frac{rG}{2}-\frac{hrG}{2}\alpha T=0 \tag{B.12}$$

Comparing the coefficients of like powers of p in (B.10) we get

$$p^0:\frac{d^2T_0}{dr^2}+\frac{Br}{4}G^2r^2=0 \tag{B.13}$$

$$p^1:\frac{d^2T_1}{dr^2}-(1+h)\frac{d^2T_0}{dr^2}-(1+h)\frac{Br}{4}G^2r^2-\frac{h}{r}\frac{dT_0}{dr}-h\frac{Br}{4}G^2\alpha r^2T_0=0 \tag{B.14}$$

Solving eqns.(B.11)-(B14) using the boundary conditions equation (B.7) and (B.8) we obtain the following results:

$$u_0(r)=\frac{G}{4}\left(1-r^2\right) \tag{B.15}$$

$$u_1(r)=h\left\{\frac{G\alpha}{576Bi}\left[12BrG^2r^2+3BrBiG^2r^2-BrBiG^2r^6-12BrG^2-2BrBiG^2\right]\right\} \tag{B.16}$$

$$T_0(r)=\frac{1}{48Bi}\left[-BrBiG^2r^4+4BrG^2+BrBiG^2\right] \tag{B.17}$$

$$T_1(r)=-h\left\{\begin{array}{l}\left[\frac{BrG^2}{144}r^4+\frac{BrG^2\alpha}{192Bi}\left(\begin{array}{l}\frac{BrBiG^2r^8}{56}-\\ \left(4BrG^2+BrBiG^2\right)\frac{r^4}{12}\end{array}\right)\right]\\ -\frac{1}{Bi}\left[\begin{array}{l}\frac{BrG^2}{36}-\frac{BrG^2\alpha}{48Bi}\left(\frac{BrBiG^2}{21}+\frac{BrG^2}{3}\right)\\ +\frac{BrBiG^2}{144}-\frac{BrG^2\alpha}{192}\left(\begin{array}{l}\frac{BrG^2}{3}+\\ \frac{11}{168}BrBiG^2\end{array}\right)\end{array}\right]\end{array}\right\} \tag{B.18)}$$

From the HAM, we have

$$u=\lim_{p\to 1}u(r)=u_0+u_1 \tag{B.19}$$

$$T=\lim_{p\to 1}T(r)=T_0+T_1 \tag{B.20}$$

Using the eqns. (B.15) and (B.16) in (B.19) and the equation (B.17) and (B.18) in the equation (B.20), we can obtain the solutions in the text (22) and (23).

Appendix-C

Determining the region of *h* for validity

The analytical solutions represented by the equation (22) and (23) contain the auxiliary parameter h which determines the convergence region and rate of approximation for the Homotopy analysis method. The velocity and temperature profiles versus the radial distance in our discussion have the region of convergence: -0.129 to -0.091 and -1.225 to -0.960, respectively. Also the region of convergence for the entropy generation rate N_S is from -0.108 to -0.099 and the total entropy generation N_T versus the Brinkman number Br has the region of convergence: -0.101 to -0.080. For the Bejan number Be, it is from -0.174 to -0.096.

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Appendix-D

Nomenclature

Symbol	**Meaning**
r	Radial distance
$\bar{x}$	Distance measured in streamwise direction
$\bar{y}$	Distance measured in normal direction
ρ	Fluid density
L	Thermal conductivity
$\bar{T}$	Fluid temperature
T_a	Ambient temperature
α	Viscosity variation parameter
$\bar{u}$	Axial velocity
$\bar{v}$	Normal velocity
U	Velocity scale
c_p	Specific heat at constant pressure.
$\bar{P}$	Pressure
h	Transfer coefficient
K	Thermal conductivity
$\bar{\mu}$	Temperature dependant viscosity
μ_0	Fluid dynamic viscosity
Pr	Prandtl number
Br	Brinkman number
Bi	Biot number
Re	Reynolds number
S^m	Entropy generation per unit volume
S^T	Total entropy generated in the pipe flow
N_S	Entropy generation number
N_T	Total entropy generated in dimensionless form
Be	Bejan number
Φ	Irreversibility distribution ratio