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# A DISCUSSION ON SSP STRUCTURE OF PAN, HELM AND CROWN GRAPHS

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**ABSTRACT**

A computer network is a system whose components are autonomous computers and other devices that are connected together usually over long physical distance. Each computer has its own operating system and there is no direct cooperation between the computers in the execution of programs. A basic feature for a system is that its components are connected together by physical communication links to transmit information according to some pattern. Moreover, it is undoubted that the power of a system is highly dependent upon the connection pattern of components in the system. A connection pattern of the components is a system is called an interconnection network, or network for short, of the system. In this paper, using the new graph class called Super Strongly Perfect (SSP), some bipartite and non bipartite based interconnection networks (Path, Helm and crown graphs) are discussed.

**Keywords:** super strongly perfect (SSP) graph, inter connection network, Pan, helm, crown graphs.

## 1. INTRODUCTION

The use of mathematics is quite interesting in every area of computer science (i.e.) in artificial intelligence, automatic control, distributed and concurrent algorithms, software development environments and tools, software architecture and design and multiprocessing etc. Concepts in Mathematics help in the design, implementation and analysis of algorithms for scientific and engineering applications. It also improves the effectiveness and applicability of existing methods and algorithms. Graph theory is one of the important areas in mathematics. There are many research papers explore the use of graphs for modelling communication networks. The graph theoretical ideas are used by various computer applications like data mining, image segmentation, clustering, image capturing, networking etc. Graph theory can be used to represent communication networks. A communications network is a network which contains a collection of terminals, links and nodes which connect to enable telecommunication between users of the terminals. Each terminal in the network must have a unique address so messages or connections can be routed to the correct recipients. The collection of addresses in the network is called the address space. Every communications network has three basic components: 1) terminals (the starting and stopping points of network), 2) processors (which provide data transmission control functions), 3) transmission channels (which help in data transmission). The communication network aims to transmit packets of data between computers, telephones, processors or other devices. The term packet refers to some roughly fixed-size quantity of data, 256 bytes or 4096 bytes. The packets are transmitted from input to output through various switches. The communication networks can be represented using the various mathematical structures which also help us to compare the various representations based on congestion, switch size and switch count. Graphs have an important application in modeling communications networks. Generally, vertices in graph represent terminals, processors and edges represent transmission channels like wires, fibers etc. through which the data flows. Thus, a data packet hops through the network from an input terminal, through a sequence of switches joined by directed edges, to an output terminal [9]. Have we ever wondered how our mail gets from our mailbox to another address? Perhaps we are fascinated with how internet traffic travels from one country to the next. Networks have been used in a variety of applications for hundreds of years. In the scope of mathematics, we can visually depict these networks better through graphs. A graph is a representation of a group or set of objects, called vertices, in which some of the vertices are connected by links, also known as edges. The study of these graphs is referred to as Graph Theory. Figure-1 and Figure-2 are examples of simple graphs. The example used in Figure-1 is known as an undirected graph, a graph in which the edges have no orientation. A more practical example of an undirected graph would be two people shaking hands. Person A is shaking hands with person B and at the same time, person B is shaking hands with person A. Figure-2 shows a directed graph, or digraph. A digraph has edges that have direction and are called arcs. Arrows on the arcs are used to show the flow from one node to another. For example, from Figure-2, vertex A can move to vertex B, but B cannot move to A. Often times, graphs will be labeled with a number on the link between nodes. This means that the graph is weighted and this number denotes the cost it takes to get from one vertex to the next. There are many topics being researched today related to Graph Theory.

Any packet can be modeled as a directed graph where nodes are the routers, arcs are subnets. Routing function for a packet equivalent to finding shortest path in the graph associated to the network (i.e.) Minimum number of hops (un weighted graph) and shortest path (weighted graph)). The star network, a computer network modelled after the star graph, is important in distributed computing.

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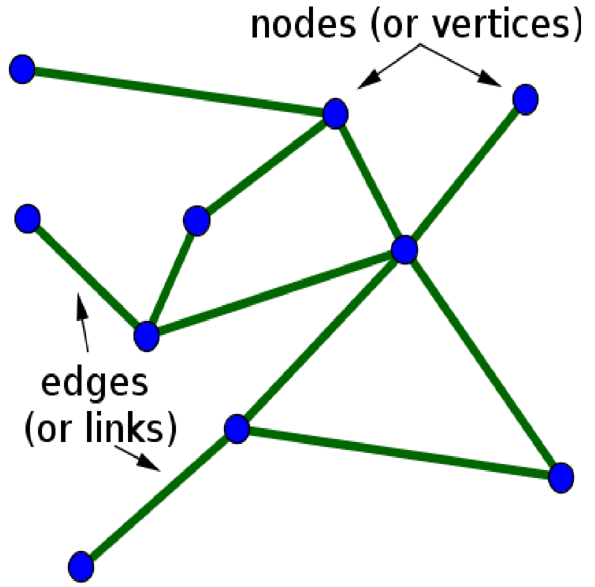


**Figure-1.** Undirected graph.

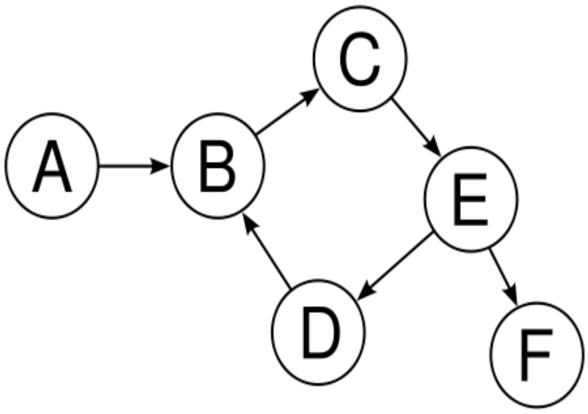


**Figure-2.** Directed graph.

An interconnection network connects the processors of a parallel and distributed system. The topology of an interconnection network for a parallel and distributed system can always be represented by a graph, where each vertex represents a processor and each edge represents a vertex-to-vertex communication link. The interconnection network plays a central role in determining the overall performance of a multicomputer system. If the network cannot provide adequate performance, for a particular application, nodes (points) will frequently be forced to wait for data to arrive. Some of the more important networks include Mesh, Torus, Rings, Hypercube, Butterfly, Benes and Cube Connected Cycles etc., [8]. It quit natural that an interconnection network may be modelled by a simple graph whose vertices represent components of the network and whose edges represent physical communication links, where directed edges represent one-way communication links and undirected edges represent two-way communication links, and the incidence function specifies a way that components of the network are interconnected by links. Such a graph is called the topological structure of the interconnection network, or network topology for short. Conversely any graph can be considered as a topological structure of some interconnection network. Topologically, graphs and interconnection networks are the same things. Thus a graph is nothing but a network. Instead of speaking a network, components, and links we speak of a graph, vertices and edges. The graph is directed or undirected, depending upon that links are one-way or two-way in the network [4]. Super strongly perfect graph is a new graph class which was defined by B. D. Acharya in 2006 and its characterization has been given as an open problem [7]. In my previous papers, various networks have been analyzed by using SSP graph. This paper investigates pan, helm and crown graphs.

## 2. BASIC CONCEPTS

In this paper, graphs are finite and simple. Let G = (V, E) be a graph where V is the vertex set and E is the edge set. A clique X is a subset of V such that G[X] is complete. A subset D of V (G) is called a dominating set if every vertex in V - D is adjacent to at least one vertex in D. A subset S of V is said to be a minimal dominating set if S-{u} is not a dominating set for any $u \in S$. Walk of a graph is an alternating sequence of vertices and edges $v_0e_1v_1e_2... v_{n-1}e_nv_n$ beginning and ending with vertices such that each edge $e_i$ is incident with $v_{i-1}$ and $v_i$. We say the walk joins $v_0$ and $v_n$ and it is called $v_0$-$v_n$ walk, $v_0$ is called the initial vertex and $v_n$ is called the terminal vertex of the walk. The above walk is also denoted by $v_0v_1v_2....v_n$, the edges of a walk being self evident and n is called the length of this walk. Walk is called a path if all the vertices are distinct. A $v_0$-$v_n$ walk is closed if $v_0 = v_n$. Closed walk $v_0v_1v_2...v_n = v_0$ in which $n \geq 3$ and $v_0, v_1, v_2, ... v_n$ are distinct is called a cycle or circuit of length n. A cycle graph of length n is denoted by $C_n$. An odd cycle is a cycle with odd number of vertices. An even cycle is a cycle with even number of vertices. In cycle, number of vertices is equal to the number of edges. Wheel Graph G is a graph with n vertices formed by connecting a single vertex to all vertices of an n-1 cycle. A bipartite graph is a graph whose vertices can be divided into two disjoint sets $V_1$ and $V_2$ (that is, $V_1$ and $V_2$ are independent sets) such that every edge connects a vertex in $V_1$ and $V_2$. A complete bipartite graph G is a bipartite graph such that for any two vertices, $v_1 \in V_1$ and $v_2 \in V_2$, $v_1v_2$ is an edge in G. The complete bipartite graph with partitions of size $|V_1| = m$ and $|V_2| = n$, is denoted as $K_{m,n}$.

## 3. SUPER STRONGLY PERFECT GRAPH

A graph G is Super Strongly Perfect (SSP) if every induced sub graph H of G possesses a minimal dominating set that meets all the maximal cliques of H. Illustrations of super strongly perfect and non-super strongly perfect graphs are given in Figure-3 and Figure-4.

**Example 1**

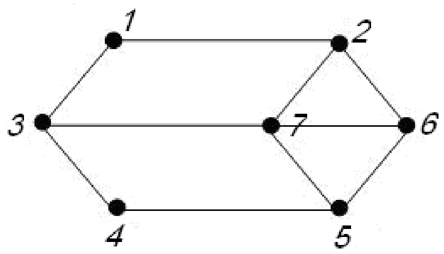


**Figure-3.** Super strongly perfect graph.

Here, {3, 7} is a minimal dominating set which meets all the maximal cliques $K_2$.

**Example 2**

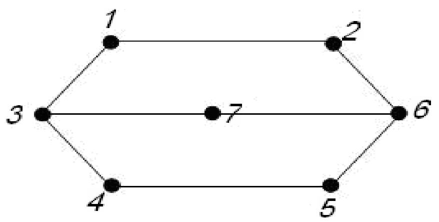


**Figure-4.** Non-Super strongly perfect graph.

Here, {1, 4, 6} is a minimal dominating set which does not meet all the maximal cliques $K_2$.

**3.1. Theorem [5]**

A graph G is super strongly perfect if and only if it does not contain an odd cycle $C_n$, $n \geq 5$ as an induced sub graph.

**3.2. Theorem [6]**

Let G be a graph with at least one maximal clique $K_n$, n = 2, 3,… G is super strongly perfect if and only if it is n-colourable.

**3.3. Proposition [6]**

Every complete k-partite graph $K_{P_1, P_2, \dots P_k}$ contains $P_1 P_2 \dots P_k$ maximal cliques $K_k$.

**3.4. Proposition [6]**

Every complete k-partite graph $K_{n,n,\dots n(k\,times)}$ contains a minimal dominating set of cardinality n.

**4. PAN GRAPH**

An n-pan graph is a graph obtained by joining a cycle graph $C_n$ to a singleton graph $K_1$ with a bridge. The special case of the 3-pan graph is sometimes known as the paw graph and the 4-pan graph as the banner graph [1]. 6-pan graph is illustrated in Figure-5.

**Example 3**

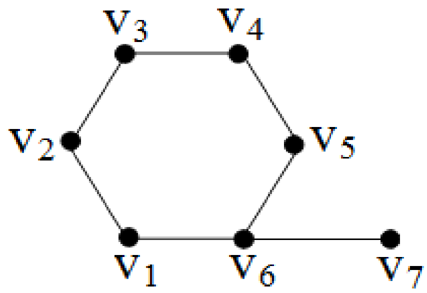


**Figure-5.** 6-Pan graph.

Here, $\{v_2, v_4, v_6\}$ is a minimal dominating set which meets all the maximal cliques of G.

**4.1. Theorem**

Every n-pan graph, $n \geq 4$ where n is even, is super strongly perfect.

**Proof**

Let G be an n-pan graph, $n \geq 4$ where n is even.
⇒From the construction of G, G does not contain an odd cycle of length atleast five as an induced sub graph.
Now, by the theorem 3.1, G is super strongly perfect.
Hence the proof.

**4.2. Theorem**

Every n-pan graph, where n is odd, $n > 3$, is non-super strongly perfect.

**Proof**

Let G be an n-pan graph, where n is odd, $n > 3$.
⇒From the construction of G, G contains an odd cycle of length atleast five as an induced sub graph.

Now, by the theorem 3.1, G is non-super strongly perfect.
Hence the proof.

**4.3. Proposition**

Let G be an n-pan graph $n \geq 4$ where n is even. G has the following properties.

1) G has n+1 maximal cliques $K_2$.
2) G is 2-colourable.
3) G has a minimal dominating set of cardinality $\left\lfloor \frac{n+1}{2} \right\rfloor$ (or) $\left\lceil \frac{n+1}{2} \right\rceil$

**Proof**

1) Let G be an n-pan graph $n \geq 4$ where n is even.
⇒G is obtained by joining $C_n$ to $K_1$.
Every $C_n$ has n edges.
Also from the construction of G, an edge is joined to $C_n$.
⇒G has n+1 maximal cliques $K_2$.
Hence the proof.
2) Let G be an n-pan graph $n \geq 4$ where n is even.
⇒By the above observation, G has n+1 maximal cliques $K_2$.

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⇒By the theorem 3.2, G is 2-colourable.
Hence the proof.

3) Let G be an n-pan graph, n ≥ 4 where n is even.
⇒G is obtained by joining $C_n$ to $K_1$.
⇒G is bipartite with n+ 1 vertices.
⇒There exists a minimal dominating set of cardinality $|V_1|$ (or) $|V_2|$.
⇒ $|V_1|$ has $\left\lfloor \frac{n+1}{2} \right\rfloor$ vertices and $|V_2|$ has $\left\lceil \frac{n+1}{2} \right\rceil$ vertices.
⇒G has a minimal dominating set of cardinality $\left\lfloor \frac{n+1}{2} \right\rfloor$ (or) $\left\lceil \frac{n+1}{2} \right\rceil$.
Hence the proof.

The above proposition is illustrated in the following Figure-6.

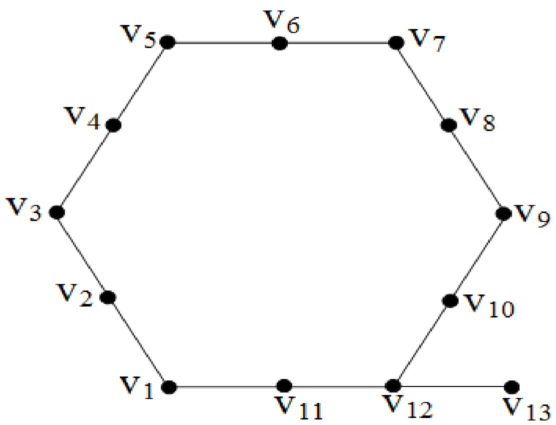


**Figure-6.** 12-Pan graph.

Here,
1) G has 13 maximal cliques $K_2$.
2) G is 2-colourable.
3) G has a minimal dominating set of cardinality 6 (or) 7.

### 4.4. Remark

Let G be a 3-pan graph then,
a) G is super strongly perfect.
b) G has only one maximal clique $K_3$.
c) G is 3-colourable.
d) G has a minimal dominating set of cardinality 1.

This remark is illustrated in the following Figure-7.

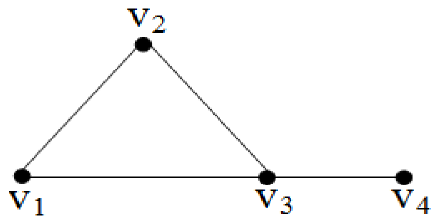


**Figure-7.** 3-Pan graph.

Here, {$v_1$, $v_4$} is a minimal dominating set which meets all the maximal cliques of G.

## 5. HELM GRAPH

The Helm Graph $H_n$ is the graph obtained from an n-wheel graph by adjoining a pendant edge at each vertex of the cycle. That is, helm graph $H_n$ is obtained by attaching a single edge and vertex to each vertex of the outer circuit of a wheel graph $W_n$. Figure-8 illustrates the helm graph $H_7$.

**Example 4**

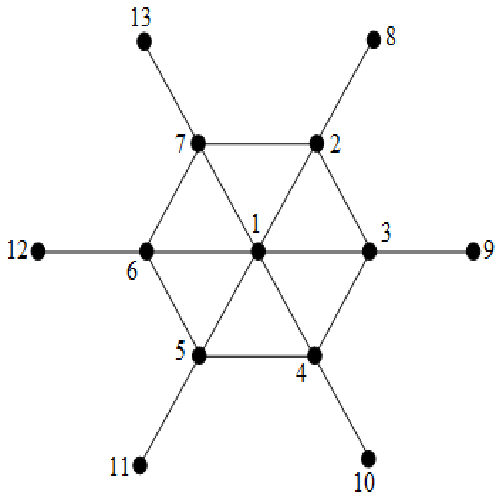


**Figure-8.** $H_7$.

Here, {1, 8, 9, 10, 11, 12, 13} is a minimal dominating set which meets all the maximal cliques of G.

### 5.1. Theorem

Every helm graph $H_n$ where n is odd, n ≥ 5, is super strongly perfect.

**Proof**

Let G be a helm graph $H_n$ where n is odd, n ≥ 5.
⇒G is constructed by an n-wheel graph by joining a pendent edge at each vertex of the cycle.

Since every odd wheel graph is super strongly perfect [3], if we join a pendent edge at each vertex of the cycle of the odd wheel graph then the resulting graph is super strongly perfect with a minimal dominating set of cardinality n vertices (i.e.,) the n-1 non adjacent vertices

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from the n-1 pendent edges and a middle vertex from the wheel graph.
⇒ G is super strongly perfect.
Hence the proof.

**5.2. Theorem**

Every helm graph $H_n$ where n is even, n ≥ 6, is non-super strongly perfect.

**Proof**

Let G be a helm graph $H_n$ where n is even, n ≥ 6.
⇒G is constructed by an n-wheel graph by joining a pendent edge at each vertex of the cycle.

Since every even wheel graph is non-super strongly perfect [3], if we join a pendent edge at each vertex of the cycle of the even wheel graph, then also the resulting graph is non-super strongly perfect.
⇒ G is non-super strongly perfect.
Hence the proof.

**5.3. Proposition**

Let G be a helm graph with odd number of vertices n, G has the following properties.
1) G contains n-1 maximal cliques $K_3$.
2) G is 3-colourable.
3) G contains a minimal dominating set of cardinality n.

**Proof**

1) Let G be a helm graph with odd number of vertices n.
⇒ G is obtained from an n-wheel graph by adjoining a pendant edge at each vertex of the outer cycle $C_{n-1}$.
⇒Any two vertices from $C_{n-1}$ with a centre single vertex give an induced $K_3$.

Since G has n-1 such vertices, G has n-1 maximal cliques $K_3$.
Hence the proof.
2) Let G be a helm graph with odd number of vertices n.
⇒ by the above part, G has n-1 maximal cliques on 3 vertices.
⇒ by the theorem 3. 2, G is 3-colourable.
Hence the proof.
3) Let G be a helm graph with odd number of vertices n.
⇒ G is formed by connecting a single vertex to all vertices of the outer cycle $C_{n-1}$.
⇒The n-1 non adjacent vertices from the n-1 pendent edges and a middle vertex from the wheel graph give a minimal dominating set which meets all the maximal cliques $K_3$.
⇒G has a minimal dominating set of cardinality n.
Hence the proof.
This observation is illustrated in the following Figure-9.

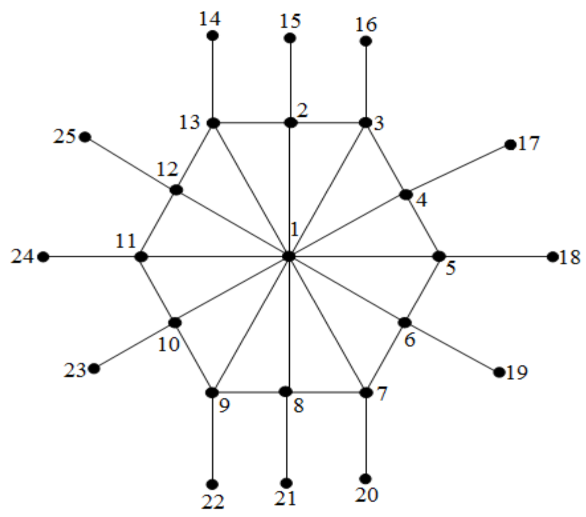


**Figure-9**. $H_{13}$.

Here,
1) G contains 12 maximal cliques $K_3$.

2) G is 3-colourable.

3) G contains a minimal dominating set of cardinality 13.

**6. CROWN GRAPH**

A Crown graph on $2n$ vertices is a graph with two sets of vertices $u_i$ and $v_i$ and with an edge from $u_i$ to $v_j$ whenever $i \neq j$. The crown graph can be viewed as a complete bipartite graph from which the edges of a perfect matching have been removed. It is denoted by $S_n^0$. The 6-vertex crown graph forms a cycle and the 8-vertex crown graph is isomorphic to the graph of a cube. A traditional rule for arranging guests at a dinner table is that men and women should alternate positions, and that no married couple should sit next to each other. The arrangements satisfying this rule, for a party consisting of $n$ married couples, can be described as the Hamiltonian cycles of a crown graph. Crown graphs can be used to show that greedy colouring algorithms behave badly in the worst case: if the vertices of a crown graph are presented to the algorithm in the order $u_0$, $v_0$, $u_1$, $v_1$, etc., then a greedy colouring uses $n$ colours, whereas the optimal number of colours is two. Crown graphs are sometimes called Johnson's graphs with notation $J_n$ [3]. Fürer uses crown graphs as part of a construction showing hardness of approximation of colouring problems [2]. A 3-crown graph is illustrated in the following Figure-10.

**Example 5**

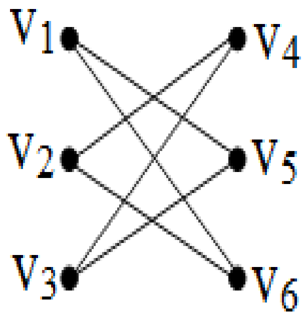


**Figure-10.** 3-Crown graph.

Here, $\{v_1, v_2, v_3\}$ is a minimal dominating set which meets all the maximal cliques of G.

**6.1. Theorem**

Every crown graph is super strongly perfect.

**Proof**

Let G be a crown graph.

⇒ From the definition of the crown graph, the removal of vertical edges (Perfect Matching) of the graph G is isomorphic to $K_{n,n}$.

Since $K_{n,n}$ is super strongly perfect, if we remove the vertical edges from $K_{n,n}$, again the graph is super strongly perfect with the same minimal dominating set of $K_{n,n}$.

⇒ G is super strongly perfect.

Hence the proof.

**6.2. Proposition**

Every crown graph has the following properties.

1) G has n(n-1) maximal cliques $K_2$.
2) G is 2-colourable.
3) G contains a minimal dominating set of cardinality n.

**Proof**

1) Let G be a crown graph.

⇒From the construction of G, the vertical edges (Perfect Matching) of $K_{n, n}$ is removed.

Also, by the proposition 3.3, $K_{n, n, n(ktimes)}$, then G contains $n^k$ maximal cliques $K_k$.

⇒$K_{n, n}$ has $n^2$ maximal cliques $K_2$.

If we remove the n-vertical edges from $K_{n, n}$, then we have $n^2$-n maximal cliques $K_2$.

⇒ G has n(n-1) maximal cliques $K_2$.

Hence the proof.

2) Let G be a crown graph.

⇒ by the previous proposition, G has n(n-1) maximal cliques, $K_2$.

⇒By theorem 3.2, G is 2-colourable.

Hence the proof.

3) Let G be a crown graph.

⇒From the construction of G, the vertical edges (Perfect Matching) of $K_{n,n}$ is removed.

Also, by proposition 3.4, $K_{n, n}$ has a minimal dominating set of cardinality n.

⇒ G has a minimal dominating set of cardinality n.

Hence the proof.

The above proposition is illustrated in the following Figure-11.

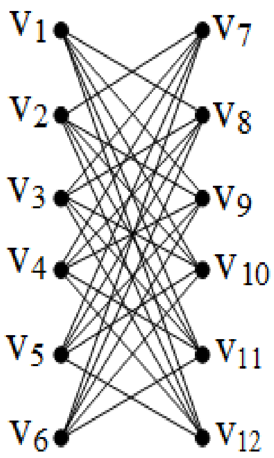


**Figure-11.** 6-Crown graph.

Here,

1) G has 30 maximal cliques $K_2$.
2) G is 2-colurable.
3) G contains a minimal dominating set of cardinality 6.

**7. CONCLUSIONS**

Artificial neural networks have been widely used in solving many issues in graph theory, such as the graph colouring problem, the graph isomorphism problem, the graph vertex coverage problems, the maximum clique and the maximum independent set problem, the planar testing problem, the graph partitioning problem, TSP problem, the Chinese postman problem and 0-1 balanced problem etc. Here, the structural problems on pan, helm and crown graphs using the new network SSP is analysed. In future, these investigations will be extended to the remaining well known networks.

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