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NULL CLINES, PHASE PLANES OF BOTH HARVESTED HOST-MORTAL COMMENSAL ECO-SYSTEM

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ABSTRACT

This paper introduces the null clines and threshold diagrams of commensalism interaction between two species. This biological model comprises commensal and host species and the host is being migrated at constant rate. Further both are consider within the limited available natural resources.

Keywords: commensalism interaction, commensal species, host species, null clines, phase planes, trajectories.

1. INTRODUCTION

The basic strategy in analyzing the phase plane is to first identify the points where solution trajectories must be exactly horizontal or exactly vertical. This will partition the phase plane into a number of regions and the general direction of trajectories in each region can be identified. From here it is often possible to see the behavior of trajectories, even without explicitly solving for them. The general concepts of modeling have been presented in the treatises of Lotka [10] and Volterra [20], Meyer [11], Cushing [5], Paul Colinvaux [12], Gause [7], Haberman [8], Pielou [13], Thompson [19], Freedman [6], Kapur [9] etc. Pattabhi Ramachryulu, Acharyulu [1-4], Seshagiri Rao and kalyani [14-18] studied the different types of interactions between two and three species with limited and unlimited recourses.

This paper presents a two species host-mortal commensal interaction with limited natural resources and when both the species are harvested at constant rates. The growth-rate equations of this model are a pair of first order non-linear ordinary differential equations. In all, nine equilibrium states are identified and the criteria for the stability of these equilibrium points have been established. Further, some threshold results are stated followed by the presentation of threshold regions for selected values of the parameters.

2. BASIC EQUATIONS

The model equations are constitute by a couple of first order non linear ordinary differential equations by following the above terminology as follows

(i). Equation for the growth rate of the Mortal Commensal species (S_1) is

$$\frac{dN_1}{dt} = a_{11}\left[-e_1N_1 - N_1^{\,2} + cN_1N_2 - H_1\right] \qquad (1)$$

(ii). Equation for the growth rate of the Host species (S_2) is

$$\frac{dN_2}{dt} = a_{22}\left[k_2N_2 - N_2^{\,2} - H_2\right] \qquad (2)$$

3. EQUILIBRIUM POINTS

The equilibrium points are given by $N_1^{'} = 0$, $N_2^{'} = 0$ are the turning points in the variation of the species N_1 and N_2 with respect to time t in the nature. The lines (straight lines and/or curves) given by $N_1^{'} = 0$ and $N_2^{'} = 0$ in the $N_1 - N_2$ plane may be referred as the threshold lines or Null clines. These lines divide the first quadrant of the $N_1 - N_2$ plane (since $N_1 \geq 0,\ N_1 \leq 0$) into regions called the threshold regions. The diagram showing the threshold lines and regions is called the threshold/phase-plane diagram. This diagram shows the direction of variations of the species around the stable/unstable equilibrium points. The diagrams presented in this paper have been drawn employing the XPPAUT software.

The system under investigation has the following nine equilibrium states and these are classified into two categories A and B.

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(A) When the harvesting rates are interdependent

(A.1) When $k_2{}^2 > 4H_2$ and $\left(c\left(\frac{k_2+\sqrt{k_2{}^2-4H_2}}{2}\right)-e_1\right)^2 = 4H_1$ (A.1)

$$E_1:\ \overline{N_1} = \frac{1}{2}\left[c\left(\frac{k_2+\sqrt{k_2{}^2-4H_2}}{2}\right)-e_1\right];\quad \overline{N_2} = \frac{k_2+\sqrt{k_2{}^2-4H_2}}{2} \tag{3}$$

(A.2) When $k_2{}^2 > 4H_2$ and $\left(c\left(\frac{k_2+\sqrt{k_2{}^2-4H_2}}{2}\right)-e_1\right)^2 > 4H_1$ (A.2)

$$E_2:\ \overline{N_1} = \frac{\left(c\left(\frac{k_2+\sqrt{k_2{}^2-4H_2}}{2}\right)-e_1\right)+\sqrt{\left(c\left(\frac{k_2+\sqrt{k_2{}^2-4H_2}}{2}\right)-e_1\right)^2-4H_1}}{2};\quad \overline{N_2} = \frac{k_2+\sqrt{k_2{}^2-4H_2}}{2} \tag{4}$$

$$E_3:\ \overline{N_1} = \frac{\left(c\left(\frac{k_2+\sqrt{k_2{}^2-4H_2}}{2}\right)-e_1\right)-\sqrt{\left(c\left(\frac{k_2+\sqrt{k_2{}^2-4H_2}}{2}\right)-e_1\right)^2-4H_1}}{2};\quad \overline{N_2} = \frac{k_2+\sqrt{k_2{}^2-4H_2}}{2} \tag{5}$$

<u>**Note:**</u> As H_1 increases and ultimately approaches to $\frac{1}{4}\left(c\left(\frac{k_2+\sqrt{k_2{}^2-4H_2}}{2}\right)-e_1\right)^2$, the two equilibrium points E_2 and E_3 coincide with E_1.

(A.3) When $k_2{}^2 > 4H_2$ and $\left(c\left(\frac{k_2-\sqrt{k_2{}^2-4H_2}}{2}\right)-e_1\right)^2 = 4H_1$ (A.3)

$$E_4:\ \overline{N_1} = \frac{1}{2}\left[c\left(\frac{k_2-\sqrt{k_2{}^2-4H_2}}{2}\right)-e_1\right];\quad \overline{N_2} = \frac{k_2-\sqrt{k_2{}^2-4H_2}}{2} \tag{6}$$

(A.4) When $k_2{}^2 > 4H_2$ and $\left(c\left(\frac{k_2-\sqrt{k_2{}^2-4H_2}}{2}\right)-e_1\right)^2 > 4H_1$ (A.4)

$$E_5:\ \overline{N_1} = \frac{\left(c\left(\frac{k_2-\sqrt{k_2{}^2-4H_2}}{2}\right)-e_1\right)+\sqrt{\left(c\left(\frac{k_2-\sqrt{k_2{}^2-4H_2}}{2}\right)-e_1\right)^2-4H_1}}{2};\quad \overline{N_2} = \frac{k_2-\sqrt{k_2{}^2-4H_2}}{2} \tag{7}$$

$$E_6:\ \overline{N_1} = \frac{\left(c\left(\frac{k_2-\sqrt{k_2{}^2-4H_2}}{2}\right)-e_1\right)-\sqrt{\left(c\left(\frac{k_2-\sqrt{k_2{}^2-4H_2}}{2}\right)-e_1\right)^2-4H_1}}{2};\quad \overline{N_2} = \frac{k_2-\sqrt{k_2{}^2-4H_2}}{2} \tag{8}$$

<u>**Note:**</u> As H_1 increases and ultimately approaches to $\frac{1}{4}\left(c\left(\frac{k_2-\sqrt{k_2{}^2-4H_2}}{2}\right)-e_1\right)^2$, the two equilibrium points E_5 and E_6 coincide with E_4.

(B) When the harvesting rates are not interdependent

(B.1) When $k_2{}^2 = 4H_2$ and $\left(\frac{ck_2}{2}-e_1\right)^2 = 4H_1$ (B.1)

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$$E_7: \overline{N_1} = \frac{1}{2}\left[\frac{ck_2}{2} - e_1\right]; \ \overline{N_2} = \frac{k_2}{2} \tag{9}$$

(B.2) When $k_2{}^2 = 4H_2$ and $\left(\frac{ck_2}{2} - e_1\right)^2 > 4H_1$ (B.2)

$$E_8: \ \overline{N_1} = \frac{\left(\frac{ck_2}{2} - e_1\right) + \sqrt{\left(\frac{ck_2}{2} - e_1\right)^2 - 4H_1}}{2}; \overline{N_2} = \frac{k_2}{2} \tag{10}$$

$$E_9: \overline{N_1} = \frac{\left(\frac{ck_2}{2} - e_1\right) - \sqrt{\left(\frac{ck_2}{2} - e_1\right)^2 - 4H_1}}{2}; \ \overline{N_2} = \frac{k_2}{2} \tag{11}$$

Note: As H_1 increases and ultimately approaches to $\frac{1}{4}\left(\frac{ck_2}{2} - e_1\right)^2$, the two equilibrium points E_8 and E_9 coincide with E_7.

The following Table-1 shows the threshold regions (I to V) around the above nine equilibrium points in the in the first quadrant (i.e., $N_1 \geq 0$, $N_2 \geq 0$) of the plane and are depending up on the signs of the derivatives of the above basic equations in that particular regions. The different regions I to IV can identify as follows:

Region I: In this region $\frac{dN_1}{dt} > 0$ an $\frac{dN_2}{dt} > 0 \Rightarrow \frac{dN_1}{dN_2} > 0$ then $N_1(t)$ is an increasing function of $N_2(t)$ and the trajectories move up and right.

Region II: Here $\frac{dN_1}{dt} > 0$ and $\frac{dN_2}{dt} < 0 \Rightarrow \frac{dN_1}{dN_2} < 0$ then $N_1(t)$ is a decreasing function of $N_2(t)$ and the trajectories move down and right.

Region III: Here $\frac{dN_1}{dt} < 0$ and $\frac{dN_2}{dt} < 0 \Rightarrow \frac{dN_1}{dN_2} > 0$ then $N_1(t)$ is an increasing function of $N_2(t)$ and the trajectories move down and left.

Region IV: Here $\frac{dN_1}{dt} < 0$ and $\frac{dN_2}{dt} > 0 \Rightarrow \frac{dN_1}{dN_2} < 0$ then $N_1(t)$ is a decreasing function of $N_2(t)$ and the trajectories move up and left.

As a brief we can see the threshold regions corresponding to the equilibrium points in the following Table-1.

Table-1.

Signs of derivatives	Threshold regions around the equilibrium points					
	E_1	E_2 and E_3	E_4	E_5 and E_6	E_7	E_8 and E_9
$\frac{dN_1}{dt} > 0$ and $\frac{dN_2}{dt} > 0$	I	I	I	I	I	I
$\frac{dN_1}{dt} > 0$ and $\frac{dN_2}{dt} < 0$	II, IV	II, IV	II, IV	II, IV	II, IV	II, IV
$\frac{dN_1}{dt} < 0$ and $\frac{dN_2}{dt} < 0$	III	III	III	III	III	III
$\frac{dN_1}{dt} < 0$ and $\frac{dN_2}{dt} > 0$	-	V	-	V	-	-

The following Table-2 shows the existing of the equilibrium points for the selected values of the model parameters a_{11}, e_1, c, H_1, a_{22}, k_2, H_2 in the basic equations.

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Table-2.

Parameters	Existing of the equilibrium points at the parameter values					
	E_1	E_2 and E_3	E_4	E_5 and E_6	E_7	E_8 and E_9
a_{11}	0.5	1.5	0.3	0.3	1.5	1.5
e_1	0.1	0.1	0.1	0.1	0.1	0.1
c	0.75	0.75	0.7	0.7	0.75	0.75
H_1	2.4447	0.5	0.4225	0.1333	0.7877	0.5
a_{22}	0.5	0.5	0.5	0.5	0.5	0.5
k_2	5	5	5	5	5	5
H_2	3	3	6	6	6.25	6.25

Category: A (When the harvesting rates are interdependent)

A. 1 The threshold diagram for equilibrium point E_1

The threshold lines/ null clines (straight lines and/or curves) divide the phase plane into four regions I, II, III and IV in the first quadrant (i.e., $N_1 \geq 0$, $N_2 \geq 0$) are shown in Figure-1. In Figure-2 we can see the direction of the field lines and the trajectories in the threshold regions around this equilibrium point E_1.

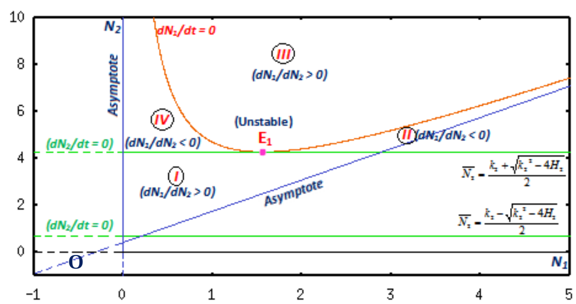

Figure-1. Threshold regions.

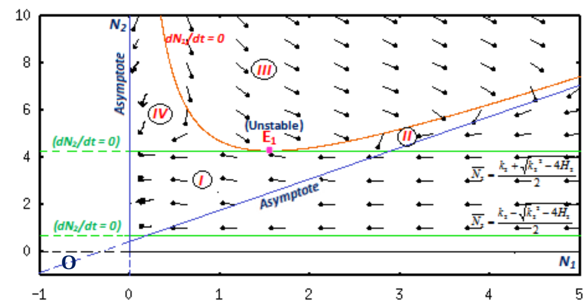

Figure-2. Threshold diagram for E_1.

A.2 The threshold diagram for equilibrium points E_2 and E_3

In this case the threshold lines divide the phase plane into five regions I, II, III, IV and V in the first quadrant (i.e., $N_1 \geq 0$, $N_2 \geq 0$) are shown in Figure-3.

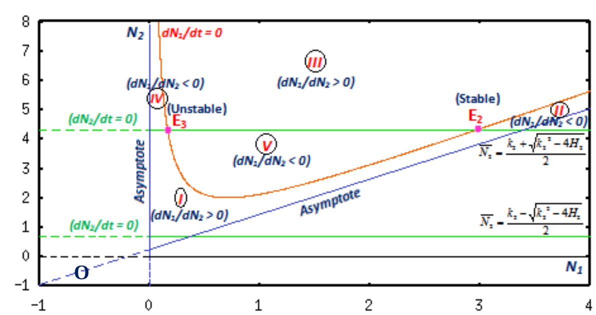

Figure-3. Threshold regions.

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The direction of the field lines and trajectories in the threshold regions for $a_{11} = 1.5, e_1 = 0.1, c = 0.75$, $H_1 = 0.5, a_{22} = 0.5, k_2 = 5$, $H_2 = 3$ are given in Figure-4.

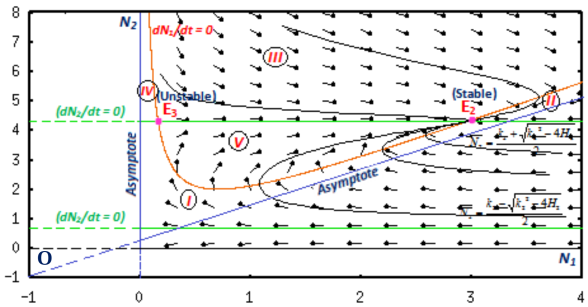

Figure-4. Threshold diagram for E_2 and E_3.

A.3 The threshold diagram for equilibrium point E_4

The null clines: straight lines and the curves divide the phase plane into four regions I, II, III and IV in the first quadrant (i.e., $N_1 \geq 0$, $N_2 \geq 0$), the trajectories and the filed lines around this equilibrium point E_4 are shown in Figures 5 and 6.

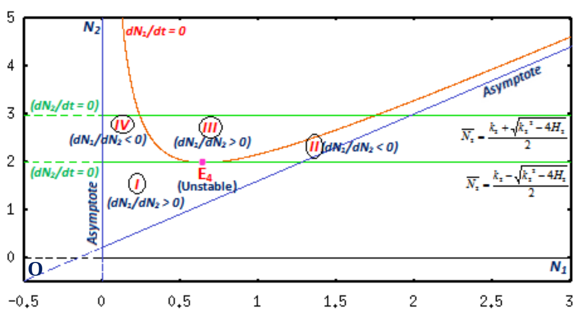

Figure-5. Threshold regions.

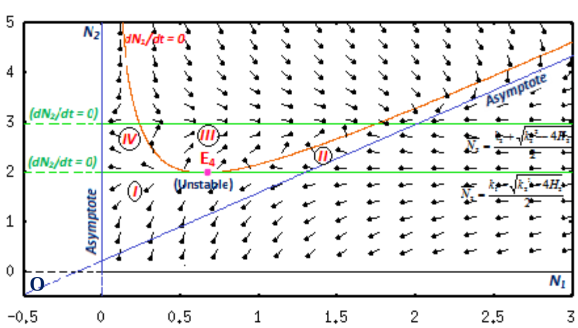

Figure-6. Threshold diagram for E_4.

A. 4 The threshold diagram for equilibrium points E_5 and E_6

In this case the threshold regions I, II, III, IV and V in the first quadrant and the threshold diagrams are shown in Figures 7 and 8.

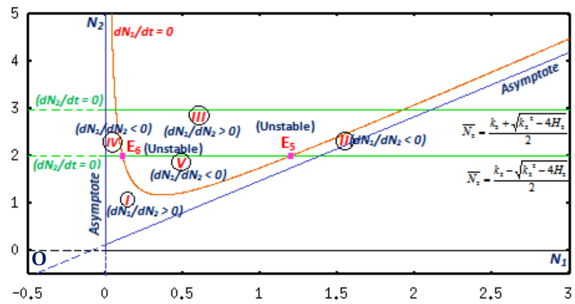

Figure-7. Threshold regions.

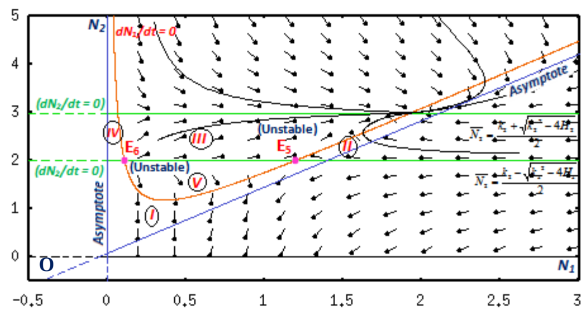

Figure-8. Threshold diagram for E_5 and E_6.

Category B (When the harvesting rates are not interdependent)

B.1 The threshold diagram for equilibrium point E_7

In this case the threshold lines given by $\frac{dN_1}{dt} = 0, \frac{dN_2}{dt} = 0$ divide the phase plane into four regions I, II, III and IV in the first quadrant (i.e., $N_1 \geq 0$, $N_2 \geq 0$) are shown in Figure-9.

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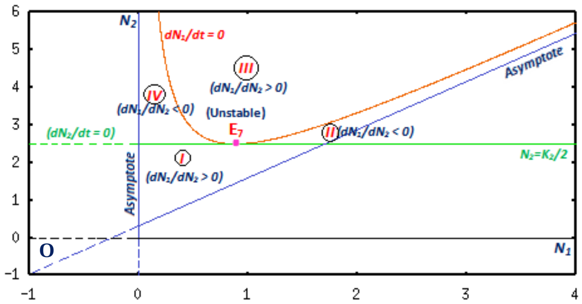

Figure-9. Threshold regions.

Figure-10 shows the direction of the field lines in the threshold regions for $a_{11}=1.5$, $e_1=0.1$, $c=0.75$, $H_1=0.7877$, $a_{22}=0.5$, $k_2=5$, $H_2=6.25$.

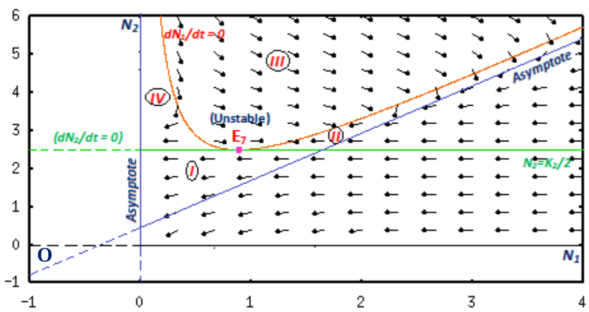

Figure-10. Threshold diagram for E_7.

B.2 The threshold diagram for equilibrium points E_8 and E_9

In this case the null clines (straight lines and curves) given by $\frac{dN_1}{dt}=0, \frac{dN_2}{dt}=0$ divide the phase plane into four regions I, II, III and IV in the first quadrant (i.e., $N_1 \geq 0$, $N_2 \geq 0$) are shown in Figure-11.

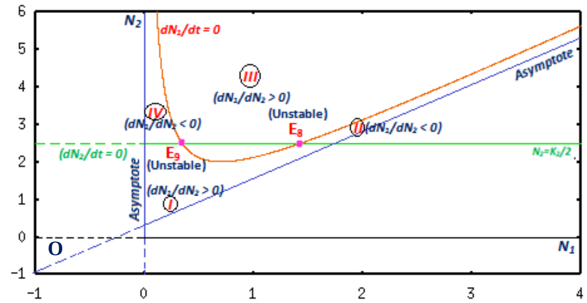

Figure-11. Threshold regions.

The direction of the field lines in the threshold regions for $a_{11}=1.5$, $e_1=0.1$, $c=0.75$, $H_1=0.5$, $a_{22}=0.5$, $k_2=5$, $H_2=6.25$.are shown in Figure-12.

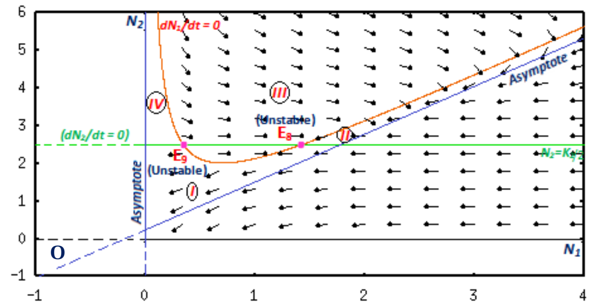

Figure-12. Threshold diagram E_8 and E_9.

4. CONCLUSIONS

The situation for the higher harvesting rates H_1 and/or H_2 of both the commensal and the host species any one of the commensal and host species extent first so that the biological system is no longer sustainable.

5. OPEN PROBLEMS

We propose the following problems to investigate in these basic model equations

a) Situations involving delayed commensalism as they occur are of interest in nature at times.

b) Immigration of the both commensal and the host at (a). Constant (b). Variable rates.

c) Constant migration of the commensal and immigration of host the species in the model equations.

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Nomenclature

$N_1(t)$, $N_2(t)$	:	The populations of the commensal and host at time t.
d_1	:	The mortal rate of the commensal species.
a_2	:	The rate of natural growth of the host species.
$a_{ii}(i=1,2)$	:	The rate of decrease of the commensal and host due to the limitations of its natural resources.
a_{12}	:	The rate of increase of the commensal due to the support given by the host.
$e_1(=d_1/a_{11})$	:	The mortality coefficient of commensal species.
$c(=a_{12}/a_{11})$	:	The coefficient of the commensal.
$k_2(=a_2/a_{22})$	:	The carrying capacity of the host species.
$h_1(=a_{11}H_1)$	:	The coefficient of harvesting /migration of the commensal species.
$h_2(=a_{22}H_2)$	:	The coefficient of harvesting /migration of the host species.
H_1, H_2	:	The harvesting/migration of the commensal and host per unit time.

The defined above variables $N_1(t)$ and $N_2(t)$ as well as all the model parameters d_1, a_2, a_{11}, a_{12}, a_{22}, k_2, e_1, c, h_1, h_2, H_1, H_2 are assumed to be non-negative constants.

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